

Twisted endoscopy from a sheaf-theoretic perspective

Aaron Christie and Paul Mezo

Abstract

The standard theory of endoscopy for real groups has two parallel formulations. The original formulation of Langlands and Shelstad relies on methods in harmonic analysis. The subsequent formulation of Adams, Barbasch and Vogan relies on sheaf-theoretic methods. The original formulation was extended by Kottwitz and Shelstad to twisted endoscopy. We extend the sheaf-theoretic formulation to the context of twisted endoscopy and provide applications for computing Arthur packets.

1 Introduction

Let G be a connected reductive algebraic group defined over a local field F and let $G(F)$ be its group of F -points. One may view the theory of endoscopy as an endeavour to make precise connections between the representations of $G(F)$ and the representations of $H(F)$, where H is a reductive group which is, in some sense, inside G . The theory is largely conjectural over nonarchimedean fields, but over archimedean fields a great deal has been worked out.

The groundbreaking results in the archimedean case are due to Langlands and Shelstad ([Lan89], [She82]). Their results were achieved largely by means of harmonic analysis—the precise connections between the representations of $G(F)$ and $H(F)$ that they proved are expressed through identities between distribution characters. An alternative perspective on endoscopy that does not derive its identities from harmonic analysis was developed by Adams, Barbasch and Vogan, culminating in the book [ABV92]. There, endoscopic identities are obtained using a duality between representations and constructible (or perverse) sheaves on varieties attached to G and H .

A natural question when presented with these two theories is to what extent the identities of one theory can be obtained from the identities of the other. Further, one could ask this question in the setting of *twisted* rather than standard endoscopy. That is exactly the line of inquiry we pursue here. Before describing the details of paper, let us give a brief tour of the ideas mentioned above.

We take for granted that the reader is somewhat familiar with the original perspective on endoscopy. For simplicity, we assume here that G is adjoint and

quasisplit over \mathbb{R} . An L -parameter is an L-homomorphism

$$\phi_G : W_{\mathbb{R}} \rightarrow {}^L G \quad (1)$$

of the real Weil group into the Langlands dual group. The local Langlands Correspondence pairs ϕ_G with an L -packet Π_{ϕ_G} , a finite set of (infinitesimal equivalence classes of) irreducible admissible representations of $G(\mathbb{R})$. If ϕ_G is an L-parameter corresponding to tempered representations, then the correspondence may be refined. In this case Shelstad has shown that there is an injective map

$$\pi \in \Pi_{\phi_G} \mapsto \tau_{\phi_G}(\pi) \quad (2)$$

to the irreducible characters of the component group of ${}^\vee G_{\phi_G}$, the centralizer in the dual group ${}^\vee G$ of the image of ϕ_G (Corollary 11.1 [She08]).

A standard *endoscopic group* H may be obtained by choosing a semisimple element $s \in {}^\vee G_{\phi_G}$ and setting the dual group ${}^\vee H$ to equal the identity component of the fixed-point subgroup ${}^\vee G^{\text{Int}(s)}$ of the inner automorphism $\text{Int}(s)$. By definition, H is a quasisplit reductive group. Avoiding some technicalities, we assume the inclusions $\phi_G(W_{\mathbb{R}}) \subset {}^L H \subset {}^L G$. We may then define the L-parameter $\phi_H : W_{\mathbb{R}} \rightarrow {}^L H$ by setting its values equal to those of ϕ_G . In the tempered case, Shelstad's endoscopic identities have the form

$$\text{Lift} \left(\sum_{\pi_H \in \Pi_{\phi_H}} \Theta_{\pi_H} \right) = \sum_{\pi \in \Pi_{\phi_G}} \tau_{\phi_G}(\pi)(\dot{s}) \Theta_{\pi} \quad (3)$$

(Corollary 11.7 [She08]). On the right, Θ_{π} is the distribution character of $\pi \in \Pi_{\phi_G}$, $\tau_{\phi_G}(\pi)$ is the character of (2), and \dot{s} is the image of s in the component group. On the left, Lift is defined through a map of orbital integrals (Theorem 6.2 [She08]). It is significant that the distribution being lifted on the left is *stable* (3 [Art89]).

There are two different ways in which one would like to extend (3). First, one would like to include nontempered representations while maintaining stability. To this end, Arthur has extended the notion of L-parameters to so-called *A-parameters*, and conjectured an extension of (3) for *A-packets* (4 [Art89]).

The other desired extension of (3) is to twisted endoscopy. The basic idea of twisted endoscopy is to allow endoscopic groups to be defined through outer automorphisms. Specifically, one would like to define H by allowing ${}^\vee H$ to equal the identity component of the fixed-point subgroup ${}^\vee G^{\vee \vartheta}$ of an outer automorphism ${}^\vee \vartheta$ of ${}^\vee G$. The foundations for twisted endoscopy were laid by Kottwitz and Shelstad in [KS99], and a partial extension of (3) has been proved for real groups ([She12], [Mez16]).

Recently, Arthur knit together these two extensions of (3) in a global context and consequently proved the nontempered version of (3) for orthogonal and symplectic groups (Theorem 2.2.1 [Art13]). His A-packets are defined using twisted endoscopy for $G = \text{GL}_N$.

We now juxtapose the theory of Adams, Barbasch and Vogan with this terse reminder of endoscopy. The underlying novelty in their theory is the

introduction of a topological space $X({}^\vee G^\Gamma)$ which reparametrizes the set of L-homomorphisms (1). This space is equipped with a ${}^\vee G$ -action and it is the ${}^\vee G$ -orbits which are in bijection with the equivalence classes of L-homomorphisms. A remarkable property of $X({}^\vee G^\Gamma)$ is that the closure relations of its ${}^\vee G$ -orbits imply relationships between the representations of different L-packets (Proposition 1.11 [ABV92]).

The characters $\tau_{\phi_G}(\pi)$ of (2) are subsumed so that the local Langlands Correspondence becomes a *bijection* between these characters and (infinitesimal equivalence classes of) irreducible representations of an extended L-packet (Theorem 1.18 [ABV92]). The L-packets are extended in that they include the usual L-packets of all inner forms of $G(\mathbb{R})$ (actually, all *strong real forms* as in Definition 1.13 [ABV92]). Putting these details aside, the extended Langlands Correspondence of [ABV92] allows one to match an irreducible representation π of $G(\mathbb{R})$ with a ${}^\vee G$ -orbit $S_{\phi_G} \subset X({}^\vee G^\Gamma)$ and a character $\tau_{\phi_G}(\pi)$. From the pair $(S_{\phi_G}, \tau_{\phi_G}(\pi))$ one may in turn define a local system on S_{ϕ_G} and a constructible sheaf on $X({}^\vee G^\Gamma)$. Thus, we have a geometric interpretation of representations. By fine-tuning this geometric interpretation to the implications of the closure relations among the ${}^\vee G$ -orbits, a duality is defined between representations and sheaves (Theorem 1.24 [ABV92]).

All of this may be carried out for an endoscopic group H , and so we may reconsider the tempered endoscopic identity (3) from this new viewpoint. The analogue of (3) is Proposition 26.7 [ABV92]. In this proposition the right-hand side differs by taking the sum over an *extended* L-packet. Similarly, the sum on the left-hand side of (3) is taken over an extended L-packet. However, the lift of the left-hand side is not defined through a map of orbital integrals. Instead, it is defined through duality together with the restriction functor on constructible sheaves.

A nontempered analogue of (3) is also proven in Theorem 26.24 [ABV92]. The theory surrounding this nontempered analogue, that of *microlocal geometry*, is quite sophisticated. We will limit ourselves to saying only that the sums in the nontempered version of (3) are taken over *micro-packets* rather than extended L-packets. The A-packets of [ABV92] are defined as a special class of these micro-packets.

One thing that is not considered in [ABV92], however, is twisted endoscopy. Our goal is to describe how an outer automorphism may be introduced into the theory of [ABV92] in a manner that reflects the theory of twisted endoscopy introduced by Kottwitz and Shelstad. We continue by outlining the contents of the paper.

In Section 2 we present the basic objects of [ABV92] and the extended local Langlands Correspondence. We assume throughout that the reader has some familiarity with [ABV92].

In Section 3 we introduce the automorphisms of the basic objects in [ABV92] and provide characterizations of automorphisms of the extended groups in terms of invariants. We define a compatibility condition between an automorphism of an extended group and an automorphism of its dual. We describe the induced action of the automorphisms on representations and on $X({}^\vee G^\Gamma)$, objects

appearing in the extended local Langlands Correspondence.

In Section 4, having obtained actions of automorphisms on objects on both sides of the extended Local Langlands Correspondence in the previous section, we formulate and prove a precise equivariance statement for the correspondence with respect to these actions (Theorem 4.2).

In the first three subsections of Section 5 we prepare for the proof of the twisted endoscopic identity. The definitions of twisted endoscopic data given in Sections 5.1 and 5.2 are amalgamations of those given in [KS99] and [ABV92]. Standard endoscopic lifting, which provides the model for twisted endoscopic lifting, is reviewed in Section 5.3. As already mentioned, it depends upon a duality between representations and sheaves, one form of which pairs irreducible representations with perverse sheaves. Standard endoscopic lifting is obtained by combining this pairing with the restriction functor on perverse sheaves.

In Section 5.4 we move on to twisted endoscopic lifting. We define *twisted* representations and *twisted* perverse sheaves, and give conditions for a natural pairing between them. This pairing is then combined with the restriction functor, just as in the standard case. After this, the only remaining complication is defining the correct objects to which one applies the lift. A geometric interpretation of these microlocal objects is given in (25.1)(j) [ABV92]. We reformulate them as virtual twisted representations in (43). These technical difficulties aside, the essential endoscopic lifting identity in its twisted form already appears in [ABV92] as Theorem 25.8. One might say without much exaggeration that twisted endoscopic lifting is proven in that theorem.

We finish in Section 6 by specializing to the context of twisted endoscopy for general linear groups as in [Art13]. An example for GL_2 is provided followed by a discussion of the computational difficulties that arise is higher rank.

Ultimately, one would like to compare the twisted endoscopic identity (46) of the final section with Arthur's twisted endoscopic identity (Theorem 2.2.1 [Art13]). In doing so, one should be able to compare the A-packets of Arthur and the A-packets of [ABV92] (*cf.* 8 [Art08]). Recently, Arancibia, Moeglin and Renard have shown that Arthur's A-packets are identical in some key cases to those of Adams-Johnson and Barbasch-Vogan ([AMR], [MR]). Undoubtedly, this indicates that Arthur's A-packets are identical to those of [ABV92].

In closing, we hope that our excursion into [ABV92] will be useful to anyone with an interest in this book. Although our treatment is not expository, we have made efforts to reveal the structure of [ABV92] without overtaxing the reader.

2 Recollections from Adams-Barbasch-Vogan

We briefly recall those objects of [ABV92] to which automorphisms may apply. Everything recalled here may be found in the first six chapters of [ABV92].

Let Γ be the Galois group $\text{Gal}(\mathbb{C}/\mathbb{R})$. Throughout G denotes a connected complex reductive algebraic group. A *weak extended group containing G* is a real Lie group G^Γ which is an extension

$$1 \rightarrow G \rightarrow G^\Gamma \rightarrow \Gamma \rightarrow 1$$

subject to the condition that every element of $G^\Gamma - G$ acts on G by conjugation as an antiholomorphic automorphism. Henceforth we fix a weak extended group G^Γ . A *strong real form* of G^Γ is an element $\delta \in G^\Gamma - G$ such that δ^2 is central and has finite order. The weak extended group G^Γ becomes an *extended group containing G* if it is endowed with the G -conjugacy class \mathcal{W} of a triple (δ, N, χ) in which δ is a strong real form, N is a maximal unipotent subgroup of G normalized by δ , and χ is a non-degenerate unitary character on the real points of N . This triple is called a *Whittaker datum*.

Let us denote the inner automorphism of $\delta \in G^\Gamma$ by $\text{Int}(\delta)$. Every strong real form δ of G^Γ defines an antiholomorphic involution $\sigma(\delta) = \text{Int}(\delta)|_G$. The fixed-point subgroup $G(\mathbb{R}, \delta) = G^{\sigma(\delta)}$ of $\sigma(\delta)$ is a real form of G . The set of such real forms constitutes an inner class of some quasisplit form of G (Proposition 2.14 [ABV92]). Two strong real forms are *equivalent* if they are G -conjugate. Equivalent strong real forms produce isomorphic real forms. However, it is sometimes also possible for inequivalent strong real forms to produce isomorphic real forms.

The (weak) extended group G^Γ may be characterized in terms of two invariants (Corollary 2.16 and Proposition 3.6 [ABV92]). The first of the two invariants is an automorphism a of the canonical based root datum $\Psi_0(G)$, which is induced from conjugation by $a(\text{ny}) \delta \in G^\Gamma - G$. To express the second invariant, we set $Z(G)$ equal to the centre of G and $\sigma_Z = \text{Int}(\delta)|_{Z(G)}$ for $a(\text{ny}) \delta \in G^\Gamma - G$. There exists an element $\delta_q \in G^\Gamma - G$ such that $G(\mathbb{R}, \delta_q)$ is a quasisplit real form and $\delta_q^2 \in Z(G)^{\sigma_Z}$. The second invariant of G^Γ is the coset $\bar{z} \in Z(G)^{\sigma_Z} / (1 + \sigma_Z)Z(G)$ of δ_q^2 (which is well-defined). The pair of invariants (a, \bar{z}) determines the weak extended group G^Γ up to isomorphism. When G^Γ is endowed with a Whittaker datum $\mathcal{W} = G \cdot (\delta_0, N, \chi)$, then $z = \delta_0^2 \in Z(G)^{\sigma_Z}$ is a canonical representative for \bar{z} . In this case z is called the second invariant of the extended group (G^Γ, \mathcal{W}) , and the pair (a, z) determines the extended group up to isomorphism.

One of our principal objects of interest is a *representation of a strong real form of G^Γ* . This is a pair (π, δ) in which δ is a strong real form of G^Γ and π is an admissible representation of $G(\mathbb{R}, \delta)$. Two such pairs (π, δ) and (π', δ') are *equivalent* if $g\delta g^{-1} = \delta'$ for some $g \in G$, and $\pi \circ \text{Int}(g^{-1})$ is infinitesimally equivalent to π' . Let $\Pi(G/\mathbb{R})$ denote the set of (equivalence classes of) irreducible representations of strong real forms of G^Γ .

We now recall the objects which are dual to G^Γ . Let ${}^\vee G$ be the dual group of G . That is to say ${}^\vee G$ is a connected complex reductive algebraic group whose canonical based root datum is dual to that of G

$$\Psi_0({}^\vee G) = {}^\vee \Psi_0(G).$$

A *weak E-group for G* is an algebraic group ${}^\vee G^\Gamma$ which is an extension

$$1 \rightarrow {}^\vee G \rightarrow {}^\vee G^\Gamma \rightarrow \Gamma \rightarrow 1.$$

Let ${}^\vee G^\Gamma$ be a weak E-group for G . Conjugation by $a(\text{ny})$ element ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$

induces an automorphism in

$$\mathrm{Aut}\Psi_0({}^\vee G) \cong \mathrm{Aut}\Psi_0(G). \quad (4)$$

We call this automorphism the *first invariant* of ${}^\vee G^\Gamma$. The weak E-group ${}^\vee G^\Gamma$ for G becomes a *weak E-group for G^Γ* if its first invariant equals the first invariant $a \in \mathrm{Aut}\Psi_0(G)$ of G^Γ under (4). From now on we assume that ${}^\vee G^\Gamma$ is a weak E-group for G^Γ . This weak E-group becomes an *E-group for G* if we endow it with a ${}^\vee G$ -conjugacy class \mathcal{D} of elements of finite order in ${}^\vee G^\Gamma - {}^\vee G$ whose inner automorphisms preserve splittings of ${}^\vee G$ (Proposition 2.11 [ABV92]).

There is a classification of (weak) E-groups in terms of first and second invariants, just as there was for extended groups (Proposition 4.4, Proposition 4.7 and Corollary 4.8 [ABV92]). The second invariant of ${}^\vee G^\Gamma$ is an element of $Z({}^\vee G)^{\theta_Z} / (1 + \theta_Z)Z({}^\vee G)$, where $\theta_Z = \mathrm{Int}({}^\vee \delta)|_{Z({}^\vee G)}$ for any ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$. The second invariant of an E-group $({}^\vee G^\Gamma, \mathcal{D})$ is the canonical representative of the previous second invariant given by the square of any element in \mathcal{D} (Definition 4.6 [ABV92]).

The dual objects to representations of strong real forms are *complete geometric parameters*. To describe these parameters we must first introduce a complex variety $X({}^\vee G^\Gamma)$, which reparametrizes the set of L-parameters pioneered by Langlands. Towards this end, let ${}^\vee \mathfrak{g}$ be the Lie algebra of ${}^\vee G$. Let $\lambda \in {}^\vee \mathfrak{g}$ be a semisimple element and set $\mathcal{O} = \mathrm{Ad}({}^\vee G) \cdot \lambda \subset {}^\vee \mathfrak{g}$. The subalgebra $\mathfrak{n}(\lambda) \subset {}^\vee \mathfrak{g}$ consists of the positive integral eigenspaces of $\mathrm{ad}(\lambda)$. Let $N(\lambda) = \exp(\mathfrak{n}(\lambda))$, $L(\lambda)$ be the centralizer in ${}^\vee G$ of λ , and $P(\lambda) = L(\lambda)N(\lambda)$. The *canonical flat through λ* is the set $\mathcal{F}(\lambda) = \mathrm{Ad}(P(\lambda)) \cdot \lambda$. The set $\mathcal{F}(\mathcal{O})$ of canonical flats which are conjugate to $\mathcal{F}(\lambda)$ encodes information about L-parameters of representations with infinitesimal character λ . This corresponds to the restriction of these L-parameters to \mathbb{C}^\times in the Weil group $W_{\mathbb{R}} = \mathbb{C}^\times \coprod j\mathbb{C}^\times$ (Proposition 5.6 [ABV92]). The set which encodes the values of these L-parameters at j is

$$\mathcal{I}(\mathcal{O}) = \{y \in {}^\vee G^\Gamma - {}^\vee G : y^2 \in \exp(2\pi i \mathcal{O})\}.$$

The set of *geometric parameters* (for \mathcal{O}) is

$$X(\mathcal{O}, {}^\vee G^\Gamma) = \{(y, \mathcal{F}(\lambda')) : \lambda' \in \mathcal{O}, y \in {}^\vee G^\Gamma - {}^\vee G, y^2 = \exp(2\pi i \lambda')\}.$$

This set is a fibre product of $\mathcal{F}(\mathcal{O})$ and $\mathcal{I}(\mathcal{O})$ which carries a natural structure of a complex algebraic variety (Proposition 6.16 [ABV92]). By definition, the set of all geometric parameters is the disjoint union

$$X({}^\vee G^\Gamma) = \coprod_{\mathcal{O}} X(\mathcal{O}, {}^\vee G^\Gamma).$$

The dual group ${}^\vee G$ acts on geometric parameters by conjugation and this action defines the notion of equivalence for geometric parameters. According to Proposition 6.17 [ABV92], the set of equivalence classes of geometric parameters is in bijection with the equivalence classes of the aforementioned L-parameters (ignoring the concept of relevance).

The local Langlands Correspondence, as originally conceived, is a bijection between (equivalence classes of) L-parameters and L-packets. Adams, Barbasch and Vogan refine and extend the local Langlands Correspondence to a bijection between (equivalence classes of) complete geometric parameters and (equivalence classes of) representations of strong real forms. They do this by supplementing each equivalence class of a geometric parameter with the representation of a finite group. To be precise, let $x = (y, \Lambda) \in X(\mathcal{O}, {}^\vee G^\Gamma)$ be a geometric parameter and $S = {}^\vee G \cdot x$ be its equivalence class. Let ${}^\vee G_x$ be the isotropy group of x and ${}^\vee G_x^{alg}$ be the preimage of this isotropy group in the universal algebraic cover

$$1 \rightarrow \pi_1({}^\vee G)^{alg} \rightarrow {}^\vee G^{alg} \rightarrow {}^\vee G \rightarrow 1. \quad (5)$$

The finite group in question is the component group

$$A_x^{loc,alg} = {}^\vee G_x^{alg} / ({}^\vee G_x^{alg})_0$$

(Definition 7.6 [ABV92]). Experts will recognize that this group is an enlargement of the usual Langlands component group of an L-parameter. It is therefore fitting to call this enlargement the *Langlands component group for x* (or S). It is important to realize that $A_x^{loc,alg}$ is abelian (page 61 [ABV92]). It therefore makes sense to identify $A_S^{loc,alg}$ with $A_x^{loc,alg}$ and speak of a representation of $A_S^{loc,alg}$.

The geometric parameter $x = (y, \Lambda)$ becomes a *(local) complete geometric parameter* if it is paired with an irreducible representation τ of $A_S^{loc,alg}$. As $A_S^{loc,alg}$ is abelian, the representation τ is actually a character. A *complete geometric parameter* is a pair of the form $(S, \tau \circ \text{Int}({}^\vee G^{alg}))$, in other words an equivalence class of a local complete geometric parameter. The set of complete geometric parameters is denoted by $\Xi({}^\vee G^\Gamma)$.

The extension of the original Langlands Correspondence for real groups takes the shape of a bijection

$$\Pi(G/\mathbb{R}) \xleftrightarrow{LLC} \Xi({}^\vee G^\Gamma) \quad (6)$$

in Theorem 1.18 [ABV92]. It is an extension in the sense that it combines the traditional Langlands correspondence for all inner forms of a quasisplit form into a single bijection.

There are two important variants of (6) in [ABV92] which we sketch briefly. Correspondence (6) is only valid for an E-group ${}^\vee G^\Gamma$ whose second invariant $z \in Z({}^\vee G)^{\theta z}$ is trivial. When z is non-trivial, rather than considering representations of a strong real form $G(\mathbb{R}, \delta)$, one must consider representations of a certain profinite cover

$$1 \rightarrow \pi_1(G)^{can} \rightarrow G(\mathbb{R}, \delta)^{can} \rightarrow G(\mathbb{R}, \delta) \rightarrow 1 \quad (7)$$

(Definition 10.3 [ABV92]) whose restrictions to $\pi_1(G)^{can}$ are related to z . The set of (equivalence classes of) these *canonical projective representations of type*

z is denoted by $\Pi^z(G/\mathbb{R})$. For non-trivial second invariant z , correspondence (6) is expressed as a bijection

$$\Pi^z(G/\mathbb{R}) \xleftrightarrow{LLC} \Xi^z({}^\vee G^\Gamma) = \Xi({}^\vee G^\Gamma) \quad (8)$$

(Theorem 10.4 [ABV92]).

One may consider a quotient Q of the algebraic fundamental group $\pi({}^\vee G)^{alg}$ in (5) and use the smaller cover

$$1 \rightarrow Q \rightarrow {}^\vee G^Q \rightarrow {}^\vee G \rightarrow 1$$

in place of ${}^\vee G^{alg}$ to define a *complete geometric parameter of type Q* (Definition 7.6 [ABV92]). The difference here is that the representation in the complete geometric parameter is a representation of a group

$$A_S^{loc,Q} = A_x^{loc,Q} = {}^\vee G_x^Q / ({}^\vee G_x^Q)_0$$

The set of (equivalence classes of) these parameters is denoted by $\Xi^z({}^\vee G^\Gamma)^Q$.

Looking at (8) one expects that $\Xi^z({}^\vee G^\Gamma)^Q$ corresponds bijectively to a subset of $\Pi^z(G/\mathbb{R})$. To say what this subset is requires more details about Q . The quotient Q is obtained from a closed subgroup K_Q

$$1 \rightarrow K_Q \rightarrow \pi_1({}^\vee G)^{alg} \rightarrow Q \rightarrow 1. \quad (9)$$

Let \hat{Q} be the character group of Q . One may view \hat{Q} as the subgroup of characters of $\pi_1({}^\vee G)^{alg}$ which are trivial on K_Q . By Lemma 10.9 (a) [ABV92], there is an isomorphism between the characters of $\pi_1({}^\vee G)^{alg}$ and the finite-order elements in $Z(G)$. Using this isomorphism, the character group \hat{Q} may be identified with a finite subgroup J of $Z(G)$. Define $\Pi^z(G, \mathbb{R})_J = \Pi^z(G/\mathbb{R})_{\hat{Q}}$ to consist of those $(\pi, \delta) \in \Pi^z(G/\mathbb{R})$ satisfying $\delta^2 \in zJ$ (z is the second invariant of (G, \mathcal{W})). Theorem 10.11 [ABV92] broadens (8) to include bijections of the shape

$$\Pi^z(G/\mathbb{R})_{\hat{Q}} \xleftrightarrow{LLC} \Xi^z({}^\vee G^\Gamma)^Q. \quad (10)$$

3 Automorphisms

The purpose of this section is to show how automorphisms of real reductive groups may be introduced into the framework of [ABV92]. We continue with the notation and assumptions of Section 2.

Definition 3.1.

An *automorphism of a weak extended group G^Γ* is a group automorphism ϑ^Γ of G^Γ whose restriction $\vartheta = \vartheta|_G$ is a holomorphic (algebraic) automorphism of G . Such an automorphism is an *automorphism of an extended group (G^Γ, \mathcal{W})* if the Whittaker datum $\mathcal{W} = G \cdot (\delta_0, N, \chi)$ is equal to $G \cdot (\vartheta^\Gamma(\delta_0), \vartheta(N), \chi \circ \vartheta^{-1})$.

3.1 Characterizations of automorphisms

The next two propositions characterize the relationship between automorphisms of G and automorphisms of G^Γ .

Proposition 3.2. *Suppose ϑ^Γ is an automorphism of a weak extended group G^Γ with invariants (a, \bar{z}) , and let $\vartheta = \vartheta|_G$. Then*

1. *The automorphism ϑ passes to an automorphism $\Psi_0(\vartheta)$ of $\Psi_0(G)$ which commutes with a .*
2. *The automorphism ϑ passes to an automorphism $\bar{\vartheta}$ of $Z(G)^{\sigma_Z}/(1+\sigma_Z)Z(G)$ satisfying $\bar{\vartheta}(\bar{z}) = \bar{z}$.*

Proof. The construction of the automorphism $\Psi_0(\vartheta)$ is well-known (see Proposition 2.11 [ABV92]). To see that it commutes with a we note that $a = \Psi_0(\sigma(\delta)) = \Psi_0(\text{Int}(\delta)|_G)$ for a(ny) $\delta \in G^\Gamma - G$ (Proposition 2.12 and Proposition 2.16 [ABV92]). It is easily computed that

$$\text{Int}(\vartheta^\Gamma(\delta))|_G = \vartheta \circ \text{Int}(\delta)|_G \circ \vartheta^{-1}.$$

Since the elements $\vartheta^\Gamma(\delta)$ and δ both belong to $G^\Gamma - G$ we have

$$a = \Psi_0(\text{Int}(\delta)|_G) = \Psi_0(\text{Int}(\vartheta^\Gamma(\delta))|_G) = \Psi_0(\vartheta \circ \text{Int}(\delta)|_G \circ \vartheta^{-1}).$$

It follows from the definitions that the expression on the right is equal to

$$\Psi_0(\vartheta) \circ \Psi_0(\text{Int}(\delta)|_G) \circ \Psi_0(\vartheta)^{-1} = \Psi_0(\vartheta) \circ a \circ \Psi_0(\vartheta)^{-1}$$

(cf. page 34 [ABV92]). This proves the first assertion. Similar arguments allow us to deduce that

$$\sigma_Z = \text{Int}(\delta)|_{Z(G)} = \vartheta \circ \text{Int}(\delta) \circ \vartheta|_{Z(G)}^{-1} = \vartheta \circ \sigma_Z \circ \vartheta|_{Z(G)}^{-1}.$$

This equation justifies the existence of the automorphism $\bar{\vartheta}$. For the final assertion, recall from Section 2 that \bar{z} is equal to the coset of δ_q^2 in $Z(G)^{\sigma_Z}/(1+\sigma_Z)Z(G)$, where $\sigma(\delta_q)$ is a quasisplit real form of G in the inner class defined by a . Let B be a Borel subgroup preserved by $\sigma(\delta_q)$ and set $\delta'_q = \vartheta^\Gamma(\delta_q)$. Then

$$\vartheta(B) = \vartheta \circ \text{Int}(\delta_q) \circ \vartheta^{-1}(\vartheta(B)) = \text{Int}(\vartheta^\Gamma(\delta_q))(\vartheta(B)) = \sigma(\delta'_q)(\vartheta(B))$$

implies that $\sigma(\delta'_q)$ is also a quasisplit form. Therefore, according to Corollary 2.16 (a) [ABV92], the element \bar{z} is also equal to the coset of $(\delta'_q)^2 = \vartheta(\delta_q^2)$ in $Z(G)^{\sigma_Z}/(1+\sigma_Z)Z(G)$. This is equivalent to $\bar{\vartheta}(\bar{z}) = \bar{z}$. \square

Proposition 3.3. *Suppose ϑ is a holomorphic automorphism of G , and G^Γ is a weak extended group for G with invariants (a, \bar{z}) . Suppose further that $\delta_q \in G^\Gamma - G$ yields a quasisplit form. If $\Psi_0(\vartheta)$ commutes with a then ϑ passes to an automorphism $\bar{\vartheta}$ of $Z(G)^{\sigma_Z}/(1+\sigma_Z)Z(G)$. If in addition $\bar{\vartheta}(\bar{z}) = \bar{z}$, then ϑ extends to an automorphism ϑ^Γ of G^Γ . Any two such extensions differ at their value on δ_q by a multiple of an element in $\ker(1+\sigma_Z)$.*

Proof. Suppose $\Psi_0(\vartheta)$ commutes with a . Then the existence of $\bar{\vartheta}$ follows as in the proof of the previous proposition. The commutativity of $\Psi_0(\vartheta)$ and a also implies

$$\Psi_0(\sigma(\delta_q)) = \Psi_0(\vartheta \circ \sigma(\delta_q) \circ \vartheta^{-1}). \quad (11)$$

Suppose $\bar{\vartheta}(\bar{z}) = \bar{z}$. By (11) and Proposition 2.12 [ABV92], the form $\sigma_q = \sigma(\delta_q)$ is equivalent to the form $\vartheta \sigma_q \vartheta^{-1}$. It follows from Proposition 2.14 [ABV92] that there exists $\delta_1 \in G^\Gamma - G$, $g \in G$ and $z_1 \in Z(G)$ such that $\sigma(\delta_1) = \vartheta \circ \sigma_q \circ \vartheta^{-1}$ and $\delta_1 = z_1 g \delta_q g^{-1}$. By replacing δ_1 with $z_1^{-1} \delta_1$ we may assume without loss of generality that z_1 is trivial and $\delta_1 = g \delta_q g^{-1}$. Let $z = \delta_q^2 = \delta_1^2$ be a coset representative of \bar{z} (Corollary 2.16 [ABV92]). Then $\bar{\vartheta}(\bar{z}) = \bar{z}$ amounts to $\vartheta(z) = z_2 \sigma_Z(z_2) z$ for some $z_2 \in Z(G)$. Set $\vartheta^\Gamma(\delta_q) = z_2 \delta_1$ so that $\vartheta^\Gamma(\delta_q)^2 = \vartheta(z)$ and $\vartheta \circ \sigma_q \circ \vartheta^{-1} = \sigma(\vartheta^\Gamma(\delta_q))$. To prove that ϑ^Γ defines an extension of ϑ to an automorphism of G^Γ we use the equations of (2.17)(b) [ABV92]. We compute

$$\begin{aligned} \vartheta^\Gamma(g_1 \delta_q g_2 \delta_q) &= \vartheta(g_1 \sigma_q(g_2) z) \\ &= \vartheta(g_1) \vartheta \circ \sigma_q \circ \vartheta^{-1}(\vartheta(g_2)) \vartheta^\Gamma(\delta_q)^2 \\ &= \vartheta^\Gamma(g_1 \delta_q) \vartheta^\Gamma(g_2 \delta_q) \end{aligned}$$

and

$$\begin{aligned} \vartheta^\Gamma(g_1 \delta_q g_2) &= \vartheta^\Gamma(g_1 \sigma_q(g_2) \delta_q) \\ &= \vartheta(g_1) \vartheta \circ \sigma_q \circ \vartheta^{-1}(\vartheta(g_2)) \vartheta^\Gamma(\delta_q) \\ &= \vartheta^\Gamma(g_1 \delta_q) \vartheta^\Gamma(g_2). \end{aligned}$$

For the final assertion, observe that if $\vartheta^\Gamma(\delta_q) = x z_2 \delta_1$ for some $x \in G$ then x must belong to $Z(G)$ for $\vartheta \circ \sigma_q \circ \vartheta^{-1} = \sigma(\vartheta^\Gamma(\delta_q))$ to remain true. Furthermore $x \sigma_Z(x) = 1$ for the required identity $\vartheta^\Gamma(\delta_q)^2 = \vartheta(z)$. \square

There is a parallel description of automorphisms for E-groups.

Definition 3.4.

Suppose ${}^\vee G^\Gamma$ is a weak E-group for G . An *automorphism* ς of ${}^\vee G^\Gamma$ is simply a holomorphic (algebraic) group automorphism. Such an automorphism is an *automorphism of an E-group* $({}^\vee G^\Gamma, \mathcal{D})$ for G if the conjugacy class \mathcal{D} is preserved by ς .

Since any algebraic automorphism of ${}^\vee G^\Gamma$ preserves the identity component ${}^\vee G$ this definition is analogous to Definition 3.1. There are also analogues of Propositions 3.2 and 3.3 with simpler proofs. The element $\delta_q \in G^\Gamma - G$ in the proof of Proposition 3.3 needs only to be replaced by an element ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$ whose inner automorphism is *distinguished* (i.e. preserves a splitting). One may then use the fact that distinguished automorphisms with respect to different splittings are $\text{Int}({}^\vee G)$ -conjugate, just as quasisplit forms in an inner class are $\text{Int}(G)$ -conjugate. The remaining details are left to the interested reader.

Proposition 3.5. *Suppose ς is an automorphism of the weak E-group ${}^\vee G^\Gamma$ for G with invariants (a, \bar{z}) . Then*

$$a = \Psi_0(\text{Int}({}^\vee \delta)|_{{}^\vee G}) = \Psi_0(\varsigma \circ \text{Int}({}^\vee \delta) \circ \varsigma|_{{}^\vee G}^{-1}) = \Psi_0(\varsigma|_{{}^\vee G}) \circ a \circ \Psi_0(\varsigma|_{{}^\vee G})^{-1} \quad (12)$$

for any ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$, and ς passes to an automorphism of $\bar{\varsigma}$ of $Z({}^\vee G)^{\theta_Z}/(1 + \theta_Z)Z({}^\vee G)$ satisfying $\bar{\varsigma}(\bar{z}) = \bar{z}$. Conversely, any automorphism of ${}^\vee G$, satisfying the above two equations for an element ${}^\vee \delta$ with distinguished inner automorphism, extends to an automorphism of ${}^\vee G^\Gamma$. Moreover any two such extensions differ by a multiple of an element in $\ker(1 + \theta_Z)$.

Example 3.6. The choice of extensions in Proposition 3.5 is parameterized by $\ker(1 + \theta_Z)$, and this kernel is related to twisting by one-cocycles (2.1 [KS99]). Suppose ς is an automorphism ${}^\vee G^\Gamma$. By Theorem 6.2.2 [Wei94],

$$H^1(\Gamma, Z({}^\vee G)) \cong \ker(1 + \theta_Z)/(1 - \theta_Z)Z({}^\vee G)$$

where Γ acts on $Z({}^\vee G)$ by θ_Z . Let $\mathbf{a} \in H^1(\Gamma, Z({}^\vee G))$ and \mathbf{a} be a one-cocycle in the cohomology class \mathbf{a} . By the above isomorphism, \mathbf{a} may be identified with $\mathbf{a}(\gamma) \in \ker(1 + \theta_Z)$, where $\gamma \in \Gamma$ is the non-trivial element. One may define another automorphism $\varsigma_{\mathbf{a}}$ of ${}^\vee G^\Gamma$ by following Proposition 3.5 (cf. page 17 [KS99]). More explicitly, let ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$ be an element such that $\text{Int}({}^\vee \delta)$ fixes a splitting. Then define

$$\varsigma_{\mathbf{a}}(g {}^\vee \delta) = \mathbf{a}(\gamma)^{-1} \varsigma(g {}^\vee \delta) \text{ and } \varsigma_{\mathbf{a}}(g) = \varsigma(g), \quad g \in {}^\vee G.$$

This definition is valuable even when ς is trivial (see Example 5.3).

We are assuming that ${}^\vee G^\Gamma$ is a weak E-group for G . This means that the groups G^Γ and ${}^\vee G^\Gamma$ have a common first invariant $a \in \text{Aut}(\Psi_0(G)) \cong \text{Aut}(\Psi_0({}^\vee G))$. It is natural to anticipate some sort of compatibility between an automorphism of G^Γ and an automorphism of ${}^\vee G^\Gamma$ stemming from the isomorphism $\text{Aut}(\Psi_0(G)) \cong \text{Aut}(\Psi_0({}^\vee G))$. Before making this explicit, let us consider an example motivated by 1.2 [Art13].

Example 3.7. Let $G = \text{GL}_N$ together with the automorphism ϑ equal to inverse-transpose composed with

$$\text{Int} \left(\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \ddots & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \right).$$

We define a weak extended group G^Γ by taking $\delta_q \in G^\Gamma - G$ to act on G by complex conjugation. This corresponds to the inner class of the split form of G . The first invariant $a = \Psi_0(\sigma(\delta_q))$ and second invariant $\bar{z} = \bar{\delta}_q^2$ are both trivial. By Proposition 3.3 (and its proof), ϑ extends to an automorphism ϑ^Γ fixing δ_q . Let $\mathcal{W} = G \cdot (\delta_q, N, \chi)$, where N is the upper-triangular unipotent subgroup and

$$\chi \left(\begin{bmatrix} 1 & a_1 & * & * \\ 0 & 1 & a_2 & * \\ \vdots & \cdots & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right) = \prod_{j=1}^{n-1} \chi'(a_j)$$

for some fixed non-trivial character χ' of \mathbb{R} . One may verify that $(\vartheta^\Gamma(\delta_q), \vartheta(N), \chi \circ \vartheta^{-1})$ is conjugate to (δ_q, N, χ) under the diagonal matrix $\text{diag}(1, -1, 1, -1, \dots)$. Consequently, ϑ^Γ is an automorphism of the extended group (G^Γ, \mathcal{W}) . A slightly cleaner arrangement of essentially the same example is to define ϑ as inverse-transpose composed with the inner automorphism of

$$\tilde{J} = \text{diag}(1, -1, 1, -1, \dots) \text{Int} \left(\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \ddots & 0 \\ 0 & 1 & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \right),$$

in which case $(\vartheta^\Gamma(\delta_q), \vartheta(), \chi \circ \vartheta^{-1})$ is actually equal to (δ_q, N, χ) (see (1.2.1) [Art13]). Let us continue with ϑ defined in this way.

The automorphism $\Psi_0(\vartheta)$ of the canonical based root datum for G transfers to an automorphism of the canonical based root datum for ${}^\vee G$ and the latter corresponds to an $\text{Int}({}^\vee G)$ -conjugacy class of algebraic automorphisms of ${}^\vee G$ (Proposition 2.11 [ABV92]). In this example $G = {}^\vee G = \text{GL}_N$, and ${}^\vee \vartheta = \vartheta$ is an automorphism in this conjugacy class.

We define ${}^\vee G^\Gamma = {}^\vee G \times \Gamma$ to be Langlands' L-group. In this case the first and second invariants of ${}^\vee G^\Gamma$ are again trivial. By Proposition 3.5, there is an automorphism ${}^\vee \vartheta^\Gamma$ of ${}^\vee G^\Gamma$ which is trivial on Γ and extends ${}^\vee \vartheta$. Let \mathcal{D} be the singleton containing the non-trivial element of Γ in the direct product. Then ${}^\vee \vartheta^\Gamma$ is an automorphism of the E-group $({}^\vee G^\Gamma, \mathcal{D})$ for G .

In this example the $\text{Int}({}^\vee G)$ -conjugacy class of automorphisms depends on a choice of splitting made in Proposition 2.11 [ABV92], and each automorphism in this class is *distinguished* in the sense that it preserves some splitting of ${}^\vee G$ (page 34 [ABV92]).¹ We say that an automorphism of ${}^\vee G^\Gamma$ is *distinguished* if its restriction to ${}^\vee G$ is distinguished. Furthermore, any automorphism in the above conjugacy class is compatible with ϑ^Γ in the following sense.

Definition 3.8.

Suppose ϑ^Γ is an automorphism of G^Γ and ${}^\vee \vartheta^\Gamma$ is an automorphism of ${}^\vee G^\Gamma$, a weak E-group for G . Then ϑ^Γ and ${}^\vee \vartheta^\Gamma$ are *compatible* if the dual action of $\Psi_0(\vartheta^\Gamma_G)$ on the root datum $\Psi_0({}^\vee G)$ is equal to that of $\Psi_0({}^\vee \vartheta^\Gamma_G)$.

Let us now reconsider the constructions of Example 3.7 more generally. Let ϑ^Γ be an automorphism of a weak extended group G^Γ and $\vartheta = \vartheta^\Gamma_G$. Then there is an $\text{Int}({}^\vee G)$ -conjugacy class of automorphisms of ${}^\vee G$ which correspond to ϑ . Any automorphism in this conjugacy class commutes with the first invariant of ${}^\vee G^\Gamma$ as in equation (12). However, Proposition 3.5 tells us that such an automorphism extends to automorphism of ${}^\vee G^\Gamma$ if and only if it also preserves the second invariant z . Suppose that one of the automorphisms in the conjugacy class preserves z . In this case *any* automorphism in the conjugacy class extends to an automorphism of ${}^\vee G^\Gamma$ by Proposition 3.5. Indeed, composition by inner

¹This suggests that perhaps the dual object to ϑ ought to be the $\text{Int}({}^\vee G)$ -conjugacy class, and not a particular representative.

automorphisms has no effect on the canonical based root datum, and has no effect on evaluation at the central element z . The distinguished automorphisms of ${}^\vee G^\Gamma$ which are compatible with ϑ^Γ are all obtained in this manner.

The order of distinguished automorphisms shall be important for endoscopy. Any distinguished automorphism of ${}^\vee G$ is of finite order modulo the centre (16.5 [Hum72]). As the following example shows, a distinguished automorphism of ${}^\vee G^\Gamma$ need not be of finite order even if its restriction to ${}^\vee G$ is.

Example 3.9. Let ${}^\vee G = \mathrm{GL}_1$, and ${}^\vee G^\Gamma = {}^\vee G \rtimes \Gamma$ with non-trivial $\gamma \in \Gamma$ acting on ${}^\vee G$ by inversion. In this case all automorphisms are vacuously distinguished. Moreover, $\ker(1 + \theta_Z) = Z({}^\vee G) = {}^\vee G$, and using Proposition 3.5 we may extend the trivial automorphism on ${}^\vee G$ to an automorphism ${}^\vee \vartheta^\Gamma$ on ${}^\vee G^\Gamma$ by choosing $z_1 \in {}^\vee G$ and defining ${}^\vee \vartheta^\Gamma({}^\vee \delta) = z_1 {}^\vee \delta$ for any ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$. It is simple to verify that ${}^\vee \vartheta^\Gamma$ is of finite order if and only if z_1 is of finite order.

Proposition 3.10. *Suppose ${}^\vee G$ is semisimple and ${}^\vee \vartheta^\Gamma$ is a distinguished automorphism of a weak E-group ${}^\vee G^\Gamma$ whose restriction ${}^\vee \vartheta$ to ${}^\vee G$ is of order m . Then ${}^\vee \vartheta^\Gamma({}^\vee \delta) = z_1 {}^\vee \delta$ for some ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$ and $z_1 \in Z({}^\vee G)$. Furthermore ${}^\vee \vartheta^\Gamma$ is of finite order if and only if $z_2 = (1 + {}^\vee \vartheta + \cdots + {}^\vee \vartheta^{m-1})(z_1)$ is of finite order.*

Proof. There exists an element ${}^\vee \delta \in {}^\vee G^\Gamma - {}^\vee G$ which preserves the same splitting ${}^\vee \vartheta$ does (Proposition 2.8 [ABV92]). It follows in turn that ${}^\vee \vartheta^\Gamma({}^\vee \delta)$ preserves this splitting and that ${}^\vee \vartheta^\Gamma({}^\vee \delta) = z_1 {}^\vee \delta$ for some $z_1 \in Z({}^\vee G)$.

It is easily seen that $({}^\vee \vartheta^\Gamma)^\ell(g {}^\vee \delta) = g {}^\vee \delta$ for all $g \in {}^\vee G$ only if m divides ℓ . The second assertion now follows from $({}^\vee \vartheta^\Gamma)^{mk}(g {}^\vee \delta) = g z_2^k {}^\vee \delta$. \square

3.2 Actions on $\Pi(G/\mathbb{R})$ and $\Xi({}^\vee G^\Gamma)$

Suppose (G^Γ, \mathcal{W}) is an extended group for G , ϑ^Γ is an automorphism thereof and $\vartheta = \vartheta^\Gamma|_G$. Suppose further that $({}^\vee G^\Gamma, \mathcal{D})$ is an E-group for G with second invariant $z \in Z({}^\vee G)^{\theta_Z}$, and that one (and hence all) of the dual automorphisms ${}^\vee \vartheta$ in the conjugacy class discussed at the end of the previous section preserves the ${}^\vee G$ -conjugacy class $\mathcal{D} \subset {}^\vee G^\Gamma$. Then ${}^\vee \vartheta$ fixes z , and by Proposition 3.5 we may extend ${}^\vee \vartheta$ to an automorphism ${}^\vee \vartheta^\Gamma$ of $({}^\vee G^\Gamma, \mathcal{D})$. By construction, the automorphisms ϑ^Γ and ${}^\vee \vartheta^\Gamma$ are compatible.

The automorphism ϑ^Γ has a straightforward action upon a representation of a strong real form (π, δ) , namely

$$\vartheta^\Gamma \cdot (\pi, \delta) = (\pi \circ \vartheta, (\vartheta^\Gamma)^{-1}(\delta)). \quad (13)$$

This action passes to the set of equivalence classes $\Pi(G/\mathbb{R})$. We say that (π, δ) is ϑ^Γ -stable if it is equivalent to $\vartheta^\Gamma \cdot (\pi, \delta)$ (Definition 2.13 [ABV92]).

There is also a straightforward action of the automorphism ${}^\vee \vartheta^\Gamma$ on a complete geometric parameter. To define this action clearly, we fix a representative $((y, \Lambda), \tau)$ for a complete geometric parameter (Section 2), where $(y, \Lambda) \in X({}^\vee G^\Gamma)$. We define ${}^\vee \vartheta^\Gamma \cdot (y, \Lambda) = ({}^\vee \vartheta^\Gamma(y), d {}^\vee \vartheta(\Lambda))$, where $d {}^\vee \vartheta$ is the differential of ${}^\vee \vartheta$. It is easily verified that ${}^\vee \vartheta^\Gamma \cdot (y, \Lambda) \in X({}^\vee G^\Gamma)$.

We now turn to τ , which is a character of the component group of the isotropy group of (y, Λ) in ${}^\vee G^{alg}$. By relating $\text{Aut}({}^\vee G)$ to $\text{Aut}(\Psi_0({}^\vee G))$ (Proposition 2.11 [ABV92]) one may show that ${}^\vee \vartheta$ has a unique lift to any finite algebraic cover of ${}^\vee G$. The collection of these lifts translates into a lift of ${}^\vee \vartheta$ to an automorphism of ${}^\vee G^{alg}$. We abusively also denote this lift by ${}^\vee \vartheta$. The image under ${}^\vee \vartheta$ of the isotropy group of (y, Λ) is equal to the isotropy group of ${}^\vee \vartheta^\Gamma \cdot (y, \Lambda)$. This leads to the definition of an action

$${}^\vee \vartheta^\Gamma \cdot ((y, \Lambda), \tau) = ({}^\vee \vartheta^\Gamma \cdot (y, \Lambda), \tau \circ {}^\vee \vartheta^{-1}) \quad (14)$$

in which the notation $\tau \circ {}^\vee \vartheta^{-1}$ is slightly abusive because of the identifications we are making for ${}^\vee \vartheta$. One may verify that this action passes to equivalence classes. It is a simple exercise to prove that this action on equivalence classes is insensitive to the choice of automorphism in $\text{Int}({}^\vee G) \cdot {}^\vee \vartheta$ and to the choice of extension ${}^\vee \vartheta^\Gamma$. In other words the action on $\Xi(G/\mathbb{R})$ derived from ϑ^Γ is canonical.

Finally, let us return to the action of ϑ^Γ on $\Pi(G/\mathbb{R})$. For the complete picture, we must consider how ϑ^Γ acts on canonical projective representations of strong real forms, *i.e.* on $\Pi^z(G/\mathbb{R})$ for non-trivial $z \in Z({}^\vee G)^{\theta^z}$ (Section 2). In order to extend (13) to this setting, one must lift ϑ to an automorphism of the canonical covering

$$1 \rightarrow \pi_1(G)^{can} \rightarrow G^{can} \rightarrow G \rightarrow 1.$$

This lifting is achieved in the same way as the lifting of ${}^\vee \vartheta$ to ${}^\vee G^{alg}$ above, except that one restricts to *distinguished* covers of G (Definition 10.1 [ABV92]) instead of finite algebraic covers. We identify ϑ with its lift to G^{can} in (13) to define the action of ϑ^Γ on $\Pi^z(G/\mathbb{R})$.

4 Twisting and the local Langlands Correspondence

We continue with the hypotheses of Section 3.2, so that we have compatible automorphisms ϑ^Γ of (G^Γ, \mathcal{W}) and ${}^\vee \vartheta^\Gamma$ of $({}^\vee G^\Gamma, \mathcal{D})$. Additionally, we have actions of ϑ^Γ on $\Pi^z(G/\mathbb{R})$, and of ${}^\vee \vartheta^\Gamma$ on $\Xi({}^\vee G^\Gamma)$. Our goal is to prove that the local Langlands Correspondence (10) is ϑ^Γ -equivariant; *i.e.* that if (π, δ) corresponds to $((y, \Lambda), \tau)$ then $\vartheta^\Gamma \cdot (\pi, \delta)$ corresponds to ${}^\vee \vartheta^\Gamma \cdot ((y, \Lambda), \tau)$.

Our proof of this theorem proceeds in two steps. In the first step we prove the theorem when G is an algebraic torus. In this case it is convenient to begin with the special case that the second invariant $z \in Z({}^\vee G)^{\theta^z}$ is trivial, and $Q = \pi_1({}^\vee G)^{alg}$, before giving the proof in general.

In the second step we make no assumptions on G and use the first step together with a classification of $\Pi^z(G/\mathbb{R})$ in terms of tori (Chapter 11 [ABV92]).

4.1 The ϑ^Γ -equivariance of the local Langlands Correspondence for tori

In this section we assume $G = T$ is an algebraic torus. The local Langlands Correspondence (8) for tori is essentially proved in Theorem 5.11 [AV92] (*cf.* Proposition 10.6 and Corollary 10.7 [ABV92]). We shall describe elements of this proof and apply our automorphisms to them in order to arrive at the desired equivariance. For the sake of simplicity we begin under the assumption that the second invariant z is trivial and $Q = \pi_1({}^\vee T)^{alg}$.

We suppose that $(\pi, \delta) \in \Pi(T/\mathbb{R})$ corresponds to $((y, \Lambda), \tau) \in \Xi(T/\mathbb{R})$ under (8) and denote this correspondence by

$$(\pi, \delta) \leftrightarrow ((y, \Lambda), \tau). \quad (15)$$

As T is abelian, there is only one inner form of T in the inner class determined by T^Γ . Thus, all representations π as above are representations of the real points $T(\mathbb{R})$ of this single inner form. This allows us to divide correspondence (15) into two separate correspondences

$$\pi \leftrightarrow (y, \Lambda) \text{ and } \delta \leftrightarrow \tau. \quad (16)$$

We shall prove the ϑ^Γ -equivariance of each of these two correspondences in turn.

Observing again that T is abelian, we see that the canonical flat Λ is just a singleton containing the differential $d\pi \in \mathfrak{t}^* \cong {}^\vee \mathfrak{t}$ ((9.3) and Proposition 10.6 [ABV92]). Let $-a$ be the inverse map composed with the automorphism of $X^*(T) \cong X_*({}^\vee T)$ obtained from the first invariant of T^Γ (or ${}^\vee T^\Gamma$). The pair $(y, d\pi)$ corresponds to a unique element $\nu \in X^*(T)/(1-a)X^*(T)$ ((4.7) [AV92]) whose values on the maximal compact subgroup of $T(\mathbb{R}) = T(\mathbb{R}, \delta)$ equal those of π . Furthermore, the pair $(\nu, d\pi)$ defines a quasicharacter of $T(\mathbb{R})$ equal to π (Proposition 3.26 [AV92]). In brief, we have correspondences

$$\pi \leftrightarrow (\nu, d\pi) \leftrightarrow (y, d\pi) \quad (17)$$

From this it is simple to show that

$$\pi \circ \vartheta \leftrightarrow (\nu \circ \vartheta, d\pi \circ d\vartheta)$$

when ν is regarded as an element of $X^*(T)$ and $d\pi$ is regarded as an element of \mathfrak{t}^* . If ν is identified with an element in $X_*({}^\vee T) \cong X^*(T)$, then $\nu \circ \vartheta$ is replaced with ${}^\vee \vartheta \circ \nu$. Similarly if $d\pi$ is identified with an element in ${}^\vee \mathfrak{t} \cong \mathfrak{t}^*$, then $d\pi \circ d\vartheta$ is replaced with $d{}^\vee \vartheta(d\pi)$ ((9.2)-(9.3) [ABV92]). With these identifications in place, the correspondences of (17) prescribe that

$$\pi \circ \vartheta \leftrightarrow ({}^\vee \vartheta \circ \nu, d{}^\vee \vartheta(d\pi)) \leftrightarrow {}^\vee \vartheta^\Gamma \cdot (y, d\pi) = {}^\vee \vartheta^\Gamma \cdot (y, \Lambda). \quad (18)$$

The middle correspondence here requires a computation which attaches a Langlands parameter to ${}^\vee \vartheta^\Gamma \cdot (y, d\pi)$ ((5.7) [ABV92]), and follows the action of ${}^\vee \vartheta^\Gamma$

through (4.6)(a)-(4.6)(b) and (4.7)(c) [AV92]. The key computation is in (4.7)(c) [AV92]. It is of the form

$$\begin{aligned} & \frac{1}{2}(d^\vee \vartheta(d\pi) + a \circ d^\vee \vartheta(d\pi)) - (d^\vee \vartheta(X) - a \circ d^\vee \vartheta(X)) \\ &= d^\vee \vartheta \left(\frac{1}{2}(d\pi + a(d\pi)) - (X - a(X)) \right) \\ &\leftrightarrow {}^\vee \vartheta \circ \nu \end{aligned}$$

for some $X \in {}^\vee \mathfrak{t}$. It uses the commutativity of a and $d^\vee \vartheta$, which itself ensues from ${}^\vee \vartheta^\Gamma$ preserving \mathcal{D} . This completes our proof of the ϑ^Γ -equivariance of the first correspondence in (16).

For the second correspondence in (16) set $x = (y, \Lambda)$ as above. We shall use the following three bijections with labelling from [ABV92],

$$\begin{aligned} \{\text{equivalence classes of strong real forms}\} &\xleftrightarrow{(9.10)(b)} T^{-\sigma, \text{fin}} / T_0^{-\sigma} \\ &\cong^{(9.10)(e)} (X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{-a} / (1-a)X_*(T) \\ &\cong^{(9.8)(c)} \hat{A}_x^{\text{loc, alg}}. \end{aligned}$$

We first describe each of the three bijections, and then trace the action of ϑ^Γ through each of them to obtain the desired correspondence between $(\vartheta^\Gamma)^{-1}(\delta)$ and $\tau \circ {}^\vee \vartheta^{-1}$.

In the first bijection $\sigma = \sigma_Z$ is the antiholomorphic involution of T defined by $\text{Int}(\delta)$. The subgroup $T^{-\sigma, \text{fin}}$ consists of elements $t \in T$ such that $t\sigma(t)$ has finite order, and $T_0^{-\sigma}$ is the identity component of the subgroup consisting of elements such that $t\sigma(t) = 1$. Let δ_0 be a strong real form occurring in \mathcal{W} . Then $\delta = t\delta_0$ for some element $t \in T^{-\sigma, \text{fin}}$. The strong real form δ corresponds to the coset $tT_0^{-\sigma}$ in the first bijection.

The second bijection is induced from the map which sends

$$\mu \in (X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q})^{-a} \cong (\mathfrak{t} \otimes \mathbb{Q})^{-a}$$

to $\exp(\pi i \mu) \in T^{-\sigma, \text{fin}}$ (Lemma 9.9 [ABV92]). Here, the map $-a$ is the inverse map composed with the action of the first invariant on $X_*(T)$. Suppose that $t = \exp(\pi i \mu)$ for such a μ (with apologies for the abusive notation for π here).

The third bijection is a consequence of $X_*(T) \cong X^*({}^\vee T)$ and its extension $X^*({}^\vee T^{\text{alg}}) \cong X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ ((9.7) [ABV92]). By the definition of the second correspondence in (15), this bijection sends μ to τ .

Let us follow the action of ϑ^Γ through these three bijections. Since ϑ^Γ preserves the conjugacy class \mathcal{D} of δ_0 , we compute

$$(\vartheta^\Gamma)^{-1}(\delta) = \vartheta^{-1}(t)(\vartheta^\Gamma)^{-1}(\delta_0) = \vartheta^{-1}(t)s\delta_0s^{-1} = \vartheta^{-1}(t)s\sigma(s^{-1})\delta_0$$

for some $s \in T$. The element $s\sigma(s^{-1})$ belongs to $T_0^{-\sigma}$ (Proposition 9.10 [ABV92]), so $(\vartheta^\Gamma)^{-1}(\delta)$ corresponds to the element $\vartheta^{-1}(t)$ under the first bijection. The

latter element corresponds to $d\vartheta^{-1}(\mu)$ under the second bijection. The third bijection sends $d\vartheta^{-1}(\mu)$ to $\tau \circ {}^\vee\vartheta^{-1}$. In summary, the second correspondence in (16) is ϑ^Γ -equivariant, and for tori we have the desired correspondence

$$\vartheta^\Gamma \cdot (\pi, \delta) = (\pi \circ \vartheta, (\vartheta^\Gamma)^{-1}(\delta)) \leftrightarrow ({}^\vee\vartheta^\Gamma \cdot (y, \Lambda), \tau \circ {}^\vee\vartheta^{-1}) = {}^\vee\vartheta^\Gamma \cdot ((y, \Lambda), \tau). \quad (19)$$

We now discuss the proof of (8) for tori allowing for the possibility of non-trivial $z \in Z({}^\vee T)^{\theta z} = {}^\vee T^{\theta z}$. This does not affect the second correspondence in (16) at all. However, the first correspondence is affected and we must consider the subspace

$${}^\vee\mathfrak{t}_\mathbb{Q} = X_*({}^\vee T) \otimes_\mathbb{Z} \mathbb{Q} \subset X_*({}^\vee T) \otimes_\mathbb{Z} \mathbb{C} = {}^\vee\mathfrak{t}.$$

This subspace may be identified with a subspace of \mathfrak{t}^* using $X^*(T) \cong X_*({}^\vee T)$. Suppose $\lambda \in {}^\vee\mathfrak{t}_\mathbb{Q}$ satisfies $\exp(2\pi i\lambda) = z$. According to Proposition 10.6 [ABV92] and Theorem 5.11 [AV92] the correspondences of (17) extend to the present setting if we allow ν to belong to $\lambda + X^*(T)$ (modulo $(1-a)X^*(T)$). Having made this adjustment, one may follow the action of ϑ^Γ through (17) as before to obtain (18) and the desired correspondence (19).

Finally, let us consider correspondence (10) for $\Xi^z({}^\vee T^\Gamma)^Q$. Recall that Q is a quotient of the algebraic fundamental group as in (9). Any element of $\Xi^z({}^\vee T^\Gamma)^Q$ may be equally regarded as an element $(x, \tau) \in \Xi^z({}^\vee T^\Gamma)$ in which the restriction of τ to

$$K_Q({}^\vee G_x^{alg})_0 / ({}^\vee G_x^{alg})_0 \subset A_x^{loc, alg}$$

is trivial. The automorphism ${}^\vee\vartheta^\Gamma$ acts on $\Xi^z({}^\vee T^\Gamma)^Q$ if and only if ${}^\vee\vartheta$ preserves this subgroup for all (x, τ) (via the lift of ${}^\vee\vartheta$). In order to ensure this property we assume that (the lift of) ${}^\vee\vartheta$ preserves K_Q . Equivalently, we assume that (the lift of) ${}^\vee\vartheta$ preserves Q or \hat{Q} . Under this assumption, the ϑ^Γ -equivariance of (19) for $\Pi^z(T/\mathbb{R}) \xrightarrow{LLC} \Xi^z({}^\vee T^\Gamma)$ restricts to

$$\Pi^z(T/\mathbb{R})_{\hat{Q}} \xrightarrow{LLC} \Xi^z({}^\vee T^\Gamma)^Q.$$

The simplicity of this statement is slightly misleading in that we are identifying the finite subgroup $J \subset T$, described at the end of Section 2, with \hat{Q} . This identification is made through the isomorphisms of (9.7)(d) and Lemma 9.9 (b) [ABV92]. It is straightforward to verify that these isomorphisms are ϑ^Γ -invariant in an appropriate sense. We close by restating the above result as a theorem.

Theorem 4.1. *Suppose T is an algebraic torus, (T^Γ, \mathcal{W}) is an extended group for T , and $({}^\vee T^\Gamma, \mathcal{D})$ is an E -group for T^Γ with second invariant z . Suppose further that ϑ^Γ and ${}^\vee\vartheta^\Gamma$ are compatible automorphisms of (T^Γ, \mathcal{W}) and $({}^\vee T, \mathcal{D})$ respectively, and that Q is a ${}^\vee\vartheta$ -stable quotient of $\pi_1({}^\vee T)^{alg}$ by a closed subgroup. Then the bijection*

$$\Pi^z(T/\mathbb{R})_{\hat{Q}} \xrightarrow{LLC} \Xi^z({}^\vee T^\Gamma)^Q$$

of Theorem 10.11 [ABV92] is ϑ^Γ -equivariant in the sense of (19).

4.2 The ϑ^Γ -equivariance of the local Langlands Correspondence for reductive groups

We return to considering correspondence (10) for arbitrary connected reductive G . As observed at the end of the previous section, the ϑ^Γ -equivariance of (10) is an immediate consequence of the ϑ^Γ -equivariance of correspondence (8) under the assumption that the lift of ${}^\vee\vartheta$ preserves Q . It suffices therefore to prove the ϑ^Γ -equivariance of (8). To this end, suppose (G^Γ, \mathcal{W}) is an extended group for G , and $({}^\vee G, \mathcal{D})$ is an E-group for G^Γ with second invariant z . We suppose further that ϑ^Γ and ${}^\vee\vartheta^\Gamma$ are compatible automorphisms of (G^Γ, \mathcal{W}) and $({}^\vee G, \mathcal{D})$ respectively.

Correspondence (8) is a combination of three bijections of equivalence classes

$$(\pi, \delta) \xleftrightarrow{(12.3)} (\delta, \Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+) \xleftrightarrow{(13.13), (10.7)} ((y, \Lambda_1), \tau_1) \xleftrightarrow{(12.9)} ((y, \Lambda), \tau) \quad (20)$$

where the labels above the arrows refer to the requisite results in [ABV92]. Our strategy for proving the ϑ^Γ -equivariance of (20) is to describe each of its three bijections and follow the action of ϑ^Γ or ${}^\vee\vartheta^\Gamma$ through each of them.

We begin with the description of the first correspondence in (20)

$$(\pi, \delta) \longleftrightarrow (\delta, \Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+). \quad (21)$$

On the left we have δ , (an equivalence class of) a strong real form, and π , an (equivalence class of an) irreducible representation of $G(\mathbb{R}, \delta)^{can}$. On the right we have (an equivalence class of) a *G-limit character* (Definition 12.1 [ABV92]). Its definition begins with a Cartan subgroup $T^\Gamma \subset G^\Gamma$ such that $\delta \in T^\Gamma - T$. In this circumstance $T(\mathbb{R}) = T(\mathbb{R}, \delta) \subset G(\mathbb{R}, \delta)$ is a Cartan subgroup in the usual sense, and has a canonical cover as in (7). There is an element $z(\rho) \in Z({}^\vee G)^{\theta z}$ defined from the half-sum of the positive roots of (G, T) , which is independent of the choice of positive system (Definition 4.9 [ABV92]). The *G-limit character* in (21) is a triple in which $\Lambda^{can} \in \Pi^{zz(\rho)}(T(\mathbb{R}))$ has differential $d\Lambda^{can} \in \mathfrak{t}^*$, and $R_{i\mathbb{R}}^+$ and $R_{\mathbb{R}}^+$ are systems of positive imaginary and real roots respectively, satisfying $\langle \alpha, d\Lambda^{can} \rangle \geq 0$ for all $\alpha \in R_{i\mathbb{R}}^*$ ((11.2) [ABV92]).

Correspondence (21) is a version of the Langlands classification for real reductive groups ([Lan89]). We will first consider a special case of this classification in which π has regular infinitesimal character and $d\Lambda^{can}$ is regular (Theorem 11.7 [ABV92]). Note however, that we are working with canonical covers, which are merely pro-algebraic and not algebraic as in Langlands' original classification. In this case, correspondence (21) may be described by first attaching to $\Lambda = (\Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+)$ a *standard representation* $M(\Lambda) = \text{Ind}(P \uparrow G(\mathbb{R}, \delta)^{can})(X_L)$. This is an induced representation in which the real parabolic subgroup $P \subset G(\mathbb{R}, \delta)^{can}$ is determined by $R_{i\mathbb{R}}^+$ and $R_{\mathbb{R}}^+$, and X_L is a relative discrete series representation of the Levi subgroup $L \subset P$ determined by $R_{i\mathbb{R}}^+$ (see page 108 [AV92]). The standard representation $M(\Lambda)$ has a unique irreducible quotient $\pi(\Lambda)$, its *Langlands quotient* (page 122 [ABV92]). Correspondence (21) is prescribed by $\pi = \pi(\Lambda)$.

Let us introduce the action of ϑ^Γ into (21). If π is replaced by $\pi \circ \vartheta$ then $\pi \circ \vartheta = \pi(\Lambda) \circ \vartheta$ is a representation of $\vartheta^{-1}(G(\mathbb{R}, \delta)) = G(\mathbb{R}, (\vartheta^\Gamma)^{-1}(\delta))$. We wish to show that $\pi(\Lambda) \circ \vartheta = \pi(\vartheta^\Gamma \cdot \Lambda)$ for some appropriate definition of $\vartheta^\Gamma \cdot \Lambda$. Looking back to (13), the obvious definition is

$$\vartheta^\Gamma \cdot \Lambda = \vartheta^\Gamma \cdot (\Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+) = (\Lambda^{can} \circ \vartheta, R_{i\mathbb{R}}^+ \circ d\vartheta, R_{\mathbb{R}}^+ \circ d\vartheta),$$

where $d\vartheta$ is the differential of ϑ , and ϑ is identified with its lift to canonical covers as necessary.

Let us consider $\Lambda^{can} \circ \vartheta$. Since $\vartheta(z) = \vartheta(\delta_0^2) = z$ for any strong real form δ_0 in \mathcal{W} , and $\vartheta(z(\rho)) = z(\rho)$ (Definition 4.9 [ABV92]), it is a formality to verify that $\Lambda^{can} \circ \vartheta \in \Pi^{zz(\rho)}(\vartheta^{-1}(T(\mathbb{R})))$ and corresponds to $X_L \circ \vartheta$. It is equally formal to verify that the parabolic subgroup $\vartheta^{-1}(P) \subset G(\mathbb{R}, (\vartheta^\Gamma)^{-1}(\delta))$ is determined by the positive root systems $R_{i\mathbb{R}}^+ \circ d\vartheta$ and $R_{\mathbb{R}}^+ \circ d\vartheta$. We deduce that $\pi(\Lambda) \circ \vartheta$ is the Langlands quotient of

$$M(\Lambda) \circ \vartheta \cong \text{Ind}(\vartheta^{-1}(P) \uparrow G(\mathbb{R}, (\vartheta^\Gamma)^{-1}(\delta))^{can})(X_L \circ \vartheta) = M(\vartheta^\Gamma \cdot \Lambda) \quad (22)$$

(cf. proof of Proposition 3.1 [Mez07]). In other words $\pi(\Lambda) \circ \vartheta = \pi(\vartheta^\Gamma \cdot \Lambda)$ and we are justified in writing

$$\vartheta^\Gamma \cdot (\pi, \delta) = (\pi(\Lambda) \circ \vartheta, (\vartheta^\Gamma)^{-1}(\delta)) \longleftrightarrow ((\vartheta^\Gamma)^{-1}(\delta), \vartheta^\Gamma \cdot \Lambda) = \vartheta^\Gamma \cdot (\delta, \Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+)$$

for correspondence (21) in the case of regular infinitesimal character.

The case of singular infinitesimal character necessitates some technical conditions on $\Lambda = (\Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+)$, defined as *final* in Definition 11.13 [ABV92]. Beyond this, the principal difference from the regular case is that $M(\Lambda)$ is now defined as $\text{Ind}(P \uparrow G(\mathbb{R}, \delta)^{can})(\Psi_{d\Lambda^{can}+\mu}^{d\Lambda^{can}} X_L)$. Here, $\Psi_{d\Lambda^{can}+\mu}^{d\Lambda^{can}}$ is a Zuckerman translation functor in which $\mu \in \mathfrak{t}^*$ is a strictly dominant weight with respect to some positive system containing $R_{i\mathbb{R}}^+$ and $R_{\mathbb{R}}^+$ (Definition 5.1 [SV80]²). The representation X_L is in the relative discrete series for the Levi subgroup L and has infinitesimal character $d\Lambda^{can} + \mu$. The representation $\Psi_{d\Lambda^{can}+\mu}^{d\Lambda^{can}} X_L$ belongs to the relative *limit* of discrete series for L . It has infinitesimal character $d\Lambda^{can}$ and its distribution character has a simple description given in Lemma 5.5 [SV80]. The ϑ^Γ -equivariance of (21) for singular infinitesimal character (Theorem 11.14 [ABV92]) proceeds as in the regular case, but hinges on the identity

$$\left(\Psi_{d\Lambda^{can}+\mu}^{d\Lambda^{can}} X_L \right) \circ \vartheta \cong \Psi_{(d\Lambda^{can}+\mu) \circ d\vartheta}^{d\Lambda^{can} \circ d\vartheta} (X_L \circ \vartheta)$$

required in the modified version of (22). This identity is an immediate consequence of a comparison of the distribution characters of both representations.

Having dealt with the first correspondence of (20), we move to the second, namely

$$(\delta, \Lambda^{can}, R_{i\mathbb{R}}^+, R_{\mathbb{R}}^+) \longleftrightarrow ((y, \Lambda_1), \tau_1). \quad (23)$$

As previously discussed, Λ^{can} is a representation of a cover of the real points $T(\mathbb{R}, \delta)$ of a Cartan subgroup $T^\Gamma \subset G^\Gamma$. Together with the additional data

²The superscript and subscript for Ψ are reversed in this reference.

$(\delta, R_{\mathbb{R}}^+, R_{\mathbb{R}}^+)$, T^Γ becomes a *based Cartan subgroup* of (G^Γ, \mathcal{W}) (Definition 13.5 [ABV92]). By Proposition 13.10 (a) [ABV92], this based Cartan subgroup is paired with a unique based Cartan subgroup of $({}^\vee G^\Gamma, \mathcal{D})$ (Definition 13.7 [ABV92]) up to conjugation. Let ${}^d T^\Gamma \subset {}^\vee G^\Gamma$ be a Cartan subgroup in the latter based Cartan subgroup. The pairing allows us to view ${}^d T^\Gamma$ as an E-group for the extended group $(T^\Gamma, T \cdot \delta)$ (Definition 13.9 [ABV92]). In this view, correspondence (23) becomes correspondence (15) (Proposition 13.12 [ABV92]). By Theorem 4.1 we may write

$$\vartheta^\Gamma \cdot (\delta, \Lambda^{can}, R_{\mathbb{R}}^+, R_{\mathbb{R}}^+) \longleftrightarrow {}^\vee \vartheta^\Gamma \cdot ((y, \Lambda_1), \tau_1).$$

We may now proceed to the final correspondence of (20) which is

$$((y, \Lambda_1), \tau_1) \longleftrightarrow ((y, \Lambda), \tau). \quad (24)$$

The proof of Theorem 12.9 [ABV92] defines the map from right to left in (24). The map from left to right is considerably simpler for Λ_1 . The canonical flat Λ_1 is an element in ${}^\vee \mathfrak{t}$ which maps to the canonical flat $\Lambda = \mathcal{F}(\Lambda_1)$ as in Section 2. It is an immediate consequence of the definitions that

$${}^\vee \vartheta^\Gamma \cdot (y, \Lambda_1) = ({}^\vee \vartheta^\Gamma(y), d {}^\vee \vartheta(\Lambda_1)) \mapsto ({}^\vee \vartheta^\Gamma(y), d {}^\vee \vartheta(\mathcal{F}(\Lambda_1))) = {}^\vee \vartheta^\Gamma \cdot (y, \Lambda).$$

As for τ_1 , it passes to a representation of a quotient of $A_{(y, \Lambda_1)}^{loc, alg}$ which is isomorphic to $A_{(y, \Lambda)}^{loc, alg}$ ((12.11)(e), Definition 12.4 and Proposition 13.12 (a) [ABV92]). In this way τ_1 maps to a representation τ of $A_{(y, \Lambda)}^{loc, alg}$. Moreover, the above isomorphism is derived from a natural inclusion of component groups ((12.11)(e) and Lemma 12.10 (c) [ABV92]) which is clearly ${}^\vee \vartheta^\Gamma$ -equivariant. In particular, the representation $\tau_1 \circ {}^\vee \vartheta^{-1}$ passes to a quotient which maps to the representation $\tau \circ {}^\vee \vartheta^{-1}$. This completes the ${}^\vee \vartheta^\Gamma$ -equivariance of (24) and also completes final step in proving the ϑ^Γ -equivariance of (20).

Theorem 4.2. *Suppose G is a connected reductive algebraic group, (G^Γ, \mathcal{W}) is an extended group for G , and $({}^\vee G^\Gamma, \mathcal{D})$ is an E-group for G^Γ with second invariant z . Suppose further that ϑ^Γ and ${}^\vee \vartheta^\Gamma$ are compatible automorphisms of (G^Γ, \mathcal{W}) and $({}^\vee G, \mathcal{D})$ respectively, and that Q is a ${}^\vee \vartheta$ -stable quotient of $\pi_1({}^\vee G)^{alg}$ by a closed subgroup. Then the bijection*

$$\Pi^z(G/\mathbb{R})_{\hat{Q}} \xleftrightarrow{LLC} \Xi^z({}^\vee G^\Gamma)^Q$$

of Theorem 10.11 [ABV92] is ϑ^Γ -equivariant.

5 Twisted endoscopy

The goal of this section is to merge the notion of endoscopy and endoscopic lifting as given in 26 [ABV92] with that of twisted endoscopy as given by Kottwitz and Shelstad ([KS99]). We maintain the same hypotheses on G^Γ , ${}^\vee G^\Gamma$, ϑ , ${}^\vee \vartheta^\Gamma$, etc. as in Section 4.2.

5.1 Twisted endoscopic data

The definition of endoscopic data in (2.1.1)-(2.1.4) [KS99] is made relative to the group G , not its dual group. By contrast, Definition 26.15 [ABV92] describes an endoscopic datum relative to any weak E-group. We follow the latter point of view, which applies to any weak E-group, despite our hypotheses.

Definition 5.1.

A *weak endoscopic datum* for $({}^\vee G^\Gamma, {}^\vee \vartheta^\Gamma, Q)$ is a pair $(s^Q, {}^\vee H^\Gamma)$ in which

1. $s^Q \in {}^\vee G^Q$ and its image s in ${}^\vee G$ is ${}^\vee \vartheta$ -semisimple (see (2.1.3) [KS99]).
2. ${}^\vee H^\Gamma \subseteq {}^\vee G^\Gamma$ is a weak E-group for some connected reductive group H .
3. ${}^\vee H^\Gamma$ is open in the fixed-point set of $\text{Int}(s) \circ {}^\vee \vartheta^\Gamma$.

An *endoscopic datum* for $({}^\vee G^\Gamma, {}^\vee \vartheta^\Gamma, Q)$ is a weak endoscopic datum as above together with an ${}^\vee H$ -conjugacy class \mathcal{D}_H of elements of finite order in ${}^\vee H^\Gamma - {}^\vee H$, whose inner automorphisms preserve splittings of ${}^\vee H$ (Proposition 2.11 [ABV92]).

Example 5.2. Let us return to Example 3.7 in which we take

$$\vartheta(x) = \tilde{J}(x^\top)^{-1} \tilde{J}^{-1}, \quad x \in \text{GL}_N$$

and Q to be trivial. This is the backdrop of [Art13] in which Arthur provides a class of examples for $s = s^Q$ in terms of a matrix $J_{O,S}$ (1.2 [Art13]). To be more specific, the matrix $J_{O,S}$ is defined in terms of a pair of non-negative integers (N_O, N'_S) satisfying $N = N_O + 2N'_S$ (pages 8-9 [Art13]), and for ${}^\vee \vartheta = \vartheta$ the endoscopic element s equals $J_{O,S}^{-1} \tilde{J}$ (page 11 [Art13]). We extend ${}^\vee \vartheta$ to ${}^\vee \vartheta^\Gamma$ as in Proposition 3.5.

If N_O is zero or odd then there is only one weak E-group ${}^\vee H^\Gamma$ satisfying requirements 2-3 of Definition 5.1. It is isomorphic to the direct product of Γ and ${}^\vee H = \text{SO}_{N_O} \times \text{Sp}_{2N'_S}$. When N_O is non-zero and even then ${}^\vee H$ remains the same. However, a non-trivial semidirect product with Γ is possible via an outer automorphism in the orthogonal group O_{N_O} , which is in the fixed-point set of $\text{Int}(s) \circ {}^\vee \vartheta^\Gamma$ (pages 10-11 [Art13]).

In any case, ${}^\vee H^\Gamma \cong {}^\vee H \rtimes \Gamma$. Let $\gamma \in \Gamma$ be the non-identity element. There exists an element $h \in {}^\vee H$ such that $\text{Int}(h, \gamma)$ fixes a splitting. One may therefore set \mathcal{D}_H equal to the ${}^\vee H$ -conjugacy class of (h, γ) . When ${}^\vee H^\Gamma \cong {}^\vee H \times \Gamma$ the element h may be chosen to equal the identity element. Thus $(J_{O,S}^{-1} \tilde{J}, {}^\vee H \rtimes \Gamma, \mathcal{D}_H)$ defines an endoscopic datum for $(\text{GL}_N, {}^\vee \vartheta^\Gamma, \{1\})$.

This procedure exhausts all possibilities for endoscopic data attached to $s = J_{O,S}^{-1} \tilde{J}$. To see this, recall from Section 2 that the isomorphism class of ${}^\vee H^\Gamma$ is equivalent to a pair of invariants (a, \bar{z}) , where $\bar{z} \in Z({}^\vee H)^{\text{Int}(h, \gamma)} / (1 + \text{Int}(h, \gamma))Z({}^\vee H)$. It is a simple exercise to show that the fixed-point set $Z({}^\vee H)^{\text{Int}(h, \gamma)}$ equals $Z({}^\vee H)$, and in turn that \bar{z} may be identified with an element in $Z({}^\vee H)$. According to Proposition 4.7 (c) [ABV92], this element \bar{z} determines a unique endoscopic datum.

Example 5.3. A general source for endoscopic data is outlined on page 24 [KS99]. Suppose we are given a Langlands parameter $\phi : W_{\mathbb{R}} \rightarrow {}^\vee G^\Gamma$ (Definition 5.2 [ABV92]), a semisimple element $s \in {}^\vee G$, and an automorphism ${}^\vee \vartheta^\Gamma$ of ${}^\vee G^\Gamma$. Suppose further that

$$\text{Int}(s) \circ {}^\vee \vartheta^\Gamma \circ \phi = \phi. \quad (25)$$

Then one may take $s^Q \in {}^\vee G^Q$ to be a lift of s and ${}^\vee H^\Gamma$ to be the semidirect product of $\phi(W_{\mathbb{R}})$ and the identity component of the fixed-point set of $\text{Int}(s) \circ {}^\vee \vartheta^\Gamma$ in ${}^\vee G$ (cf. (26.16) [ABV92]). One may take \mathcal{D}_H to be the conjugacy class of $\phi(\gamma)$, where $\gamma \in \Gamma$ is non-trivial.

We highlight a special case related to Example 3.6. Suppose ς is trivial in that example. Set ${}^\vee \vartheta^\Gamma = \varsigma_{\mathbf{a}}$ for the one-cocycle \mathbf{a} , and set $s = 1$. Then (25) reads as $\phi = \mathbf{a} \cdot \phi$. This case pertains to L-packets fixed by a central character attached to \mathbf{a} . We note that this is more restrictive than the twisting of 2 [KS99], where \mathbf{a} may be taken in $H^1(W_{\mathbb{R}}, Z({}^\vee G))$ not just $H^1(\Gamma, Z({}^\vee G))$. For this reason our definition of twisted endoscopic data does not accommodate twisting by arbitrary quasicharacters.

The next definition of an endoscopic group is identical to (26.17)(b) [ABV92] if one looks past the twisting data.

Definition 5.4.

Suppose $(s^Q, {}^\vee H^\Gamma, \mathcal{D}_H)$ is an endoscopic datum for $({}^\vee G^\Gamma, {}^\vee \vartheta^\Gamma, Q)$. Then an *extended endoscopic group* is a pair $(H^\Gamma, \mathcal{W}_H)$ in which H^Γ is the extended group whose first invariant equals that of ${}^\vee H^\Gamma$ and whose second invariant is trivial (see Section 2).

Example 5.5. We continue with the endoscopic datum $(J_{O,S}^{-1} \tilde{J}, {}^\vee H \rtimes \Gamma, \mathcal{D}_H)$ of Example 5.2. The first portion of its extended endoscopic group is of the form $H^\Gamma = H \rtimes \Gamma$. Here, $H = \text{SO}_{N_0} \times \text{SO}_{2N'_S+1}$ when N_0 is non-zero and even, and $H = \text{Sp}_{N_0-1} \times \text{SO}_{2N'_S+1}$ when N_0 is odd. The semidirect product here is determined by the semidirect product in ${}^\vee H \rtimes \Gamma$, since both semidirect products are determined by the first invariants.

The choice of a Whittaker datum \mathcal{W}_H runs along the same lines as the choice of \mathcal{D}_H in Example 5.2. One may choose an element $h \in H$ such that (h, γ) is a strong real form which preserves a Borel subgroup B (Proposition 2.7 and Proposition 2.14 [ABV92]). We may set \mathcal{W}_H equal to the H -conjugacy class of $((h, \gamma), N, \chi)$, where N is the commutator subgroup of B and χ is any non-degenerate unitary character of the real points $N(\mathbb{R})$. The proof of the uniqueness of this Whittaker datum follows the proof of the uniqueness of \mathcal{D}_H in Example 5.2 (Corollary 2.16 and Proposition 3.6 [ABV92]).

We record the natural definition of equivalence of endoscopic data following (26.15) [ABV92] and (2.1.6) [KS99].

Definition 5.6.

Two endoscopic data $(s^Q, {}^\vee H^\Gamma, \mathcal{D}_H)$ and $(s_1^Q, {}^\vee H_1^\Gamma, \mathcal{D}_{H_1})$ for $({}^\vee G^\Gamma, {}^\vee \vartheta^\Gamma, Q)$ are *equivalent* if there exists $g^Q \in {}^\vee G^Q$ such that

1. $g^\vee H_1^\Gamma g^{-1} = {}^\vee H^\Gamma$
2. $g\mathcal{D}_{H_1}g^{-1} = \mathcal{D}_H$
3. $g^Q s_1^Q {}^\vee\vartheta(g^Q)^{-1} \in s^Q (Z({}^\vee H)^{\theta_Z, Q})_0$

Here, $g \in {}^\vee G$ is the image of g^Q , ${}^\vee\vartheta$ is the unique lift of ${}^\vee\vartheta^\Gamma$ to an automorphism of ${}^\vee G^Q$, and $(Z({}^\vee H)^{\theta_Z, Q})_0$ is the identity component of the preimage of $Z({}^\vee H)^{\theta_Z}$ in ${}^\vee G^Q$.

5.2 Endoscopic lifting

We begin with a description of the setting for twisted endoscopic lifting. We suppose that

- ϑ^Γ is a distinguished automorphism of an extended group (G^Γ, \mathcal{W}) such that $\vartheta = \vartheta|_G^\Gamma$ is of finite order.
- ${}^\vee\vartheta^\Gamma$ is a distinguished automorphism of an E-group $({}^\vee G, \mathcal{D})$ for G , which is compatible with ϑ^Γ and is of finite order.
- Q is a quotient of $\pi_1({}^\vee G)^{alg}$ by a closed subgroup.
- $(s^Q, {}^\vee H, \mathcal{D}_H)$ is an endoscopic datum for $({}^\vee G^\Gamma, {}^\vee\vartheta^\Gamma, Q)$.
- (H, \mathcal{W}_H) is an extended endoscopic group for $(s^Q, {}^\vee H, \mathcal{D}_H)$.

The assumptions of finite order are new to this section and deserve some scrutiny. What is actually at issue here is that the restriction of ϑ to the centre $Z(G)$ must be of finite order. If an automorphism has this property, then it may be composed with an inner automorphism so that it fixes a splitting and is of finite order ((16.5) [Hum72]). Furthermore, the map $\vartheta \mapsto \Psi_0(\vartheta)$, used in Proposition 3.3 and the constructions of Example 3.7, is insensitive to composition with inner automorphisms. One may therefore make such an adjustment without affecting the property of being an automorphism of (G^Γ, \mathcal{W}) and without changing ${}^\vee\vartheta^\Gamma$. In addition, such an adjustment has no effect on the actions on the equivalence classes in $\Pi^z(G/\mathbb{R})$.

This leaves us with the question of why one should assume that ϑ is of finite order. It is clear from the discussion following the definition of compatibility (Definition 3.8) that ϑ is of finite order if and only if the distinguished automorphism ${}^\vee\vartheta = {}^\vee\vartheta|_{{}^\vee G}^\Gamma$ is of finite order. The latter condition is a fundamental requirement in the machinery of 25 [ABV92] employed in endoscopic lifting. This is ultimately the justification for the finiteness assumption.

Although the automorphism ${}^\vee\vartheta^\Gamma$ is assumed to be of finite order, the automorphism $\text{Int}(s) \circ {}^\vee\vartheta^\Gamma$ used in the definition of endoscopic data need not be. This also gets in the way of using 25 [ABV92]. We may as well take care of this problem now.

This problem occurs in standard endoscopy too and may be circumvented by replacing s^Q with an element $(s^Q)'$ of finite order ((26.21) [ABV92]). This

circumvention is made possible by Lemma 26.20 [ABV92], which relies on the semisimplicity of s . We need an analogue of this semisimplicity for $\text{Int}(s) \circ {}^\vee\vartheta^\Gamma$.

Lemma 5.7. *$(s, {}^\vee\vartheta^\Gamma)$ is a semisimple element in the algebraic group ${}^\vee G^\Gamma \rtimes \langle {}^\vee\vartheta^\Gamma \rangle$.*

Proof. The obvious semidirect product ${}^\vee G^\Gamma \rtimes \langle {}^\vee\vartheta^\Gamma \rangle$ is an algebraic group as ${}^\vee G^\Gamma$ is an algebraic group, ${}^\vee\vartheta^\Gamma$ is an algebraic morphism, and ${}^\vee\vartheta^\Gamma$ is of finite order. The first property of Definition 5.1 is equivalent to $(s, {}^\vee\vartheta^\Gamma)$ being a semisimple element in ${}^\vee G^\Gamma \rtimes \langle {}^\vee\vartheta^\Gamma \rangle$ (7 [Ste68]). \square

We may now apply Lemma 26.20 [ABV92] to the disconnected algebraic group ${}^\vee G^\Gamma \rtimes \langle {}^\vee\vartheta^\Gamma \rangle$ and the semisimple element $(s, {}^\vee\vartheta^\Gamma)$ to conclude that there is an element $(s', {}^\vee\vartheta^\Gamma) \in (s, {}^\vee\vartheta^\Gamma) Z({}^\vee H^\Gamma)_0$ of finite order, and that ${}^\vee H^\Gamma$ is open in the fixed-point set of $\text{Int}(s') \circ {}^\vee\vartheta^\Gamma$. In particular, $\text{Int}(s') \circ {}^\vee\vartheta^\Gamma$ is a finite-order automorphism and $s' \in sZ({}^\vee H)_0^{\theta_Z}$. There exists a lift $(s^Q)' \in {}^\vee G^Q$ of s' such that $(s^Q)' \in s^Q(Z({}^\vee H)^{\theta_Z, Q})_0$ and by Definition 5.6 $((s^Q)', {}^\vee H^\Gamma, \mathcal{D}_H)$ is an equivalent endoscopic datum with the desired finiteness property (cf. (26.21) [ABV92]). We may and shall assume from now on that $s^Q = (s^Q)'$.

Calling to mind the setting for this section once more, let

$$\epsilon : {}^\vee H^\Gamma \rightarrow {}^\vee G^\Gamma$$

be the inclusion map. The map ϵ induces several other maps on sets of objects we have recalled in Section 2. For example there is the map $X(\epsilon) : X({}^\vee H^\Gamma) \rightarrow X({}^\vee G^\Gamma)$ (Corollary 6.21 [ABV92]) defined by

$$X(\epsilon)(y, \Lambda) = (\epsilon(y), \mathcal{F} \circ d\epsilon(\Lambda)), \quad (y, \Lambda) \in X({}^\vee H^\Gamma). \quad (26)$$

There is also the map $\epsilon^\bullet : {}^\vee H^{Q_H} \rightarrow {}^\vee G^Q$ ((5.14)(c) [ABV92]) in which $Q_H = Q \cap ({}^\vee H^Q)_0$ ((26.1)(c) [ABV92]). The pair of maps $(X(\epsilon), \epsilon^\bullet)$ transfers the ${}^\vee H^{Q_H}$ -action on $X({}^\vee H^\Gamma)$ to the ${}^\vee G^Q$ -action on $X({}^\vee G^\Gamma)$, and therefore induces a map of the corresponding orbits ((7.17)(c) [ABV92]). Similarly, the pair $(X(\epsilon), \epsilon^\bullet)$ induces homomorphisms between isotropy subgroups and their component groups

$$A^{loc}(\epsilon) : A_{(y, \Lambda)}^{loc, Q_H} \rightarrow A_{X(\epsilon)(y, \Lambda)}^{loc, Q}$$

((7.19)(c) [ABV92]).

5.3 Standard endoscopic lifting

To explain endoscopic lifting, we must now describe how ϵ behaves on the level of sheaves on $X({}^\vee G^\Gamma)$. We will assume that the reader has some familiarity with constructible and perverse sheaves, and review some important facts provided in [ABV92].

The extension by zero map ((7.10)(c) [ABV92])

$$\mu : \Xi({}^\vee G^\Gamma)^Q \rightarrow \text{Ob } \mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$$

yields a bijection between the complete geometric parameters of type Q and the irreducible ${}^\vee G^Q$ -equivariant constructible sheaves of complex vector spaces on $X({}^\vee G^\Gamma)$. The perverse extension map ((7.10)(d) [ABV92], Definition 1.4.22 [BBD82])

$$P : \Xi({}^\vee G^\Gamma)^Q \rightarrow \text{Ob } \mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q)$$

yields a bijection between the complete geometric parameters of type Q and the irreducible ${}^\vee G^Q$ -equivariant perverse sheaves of complex vector spaces on $X({}^\vee G^\Gamma)$. Let $K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ and $K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ denote the Grothendieck groups of the categories $\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ and $\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ respectively. There is an isomorphism $\chi : K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q) \rightarrow K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ defined by

$$\chi(P) = \sum_i (-1)^i H^i P, \quad P \in \text{Ob } \mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q) \quad (27)$$

(Lemma 7.8 [ABV92]) which allows one to identify the two Grothendieck groups. Correspondence (10) furnishes a natural perfect pairing

$$\langle \cdot, \cdot \rangle : K\Pi^z(G/\mathbb{R})_{\hat{Q}} \times K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q) \rightarrow \mathbb{Z} \quad (28)$$

with the Grothendieck group of the category of finite length representations of strong real forms of G of type \hat{Q} (Definition 15.11 [ABV92]). In essence, the irreducible constructible sheaves $\mu(\xi)$ are paired with characters of standard representations $M(\xi)$ for $\xi \in \Xi({}^\vee G^\Gamma)^Q$. This pairing is equivalent to another perfect pairing, which we abusively denote the same way,

$$\langle \cdot, \cdot \rangle : K\Pi^z(G/\mathbb{R})_{\hat{Q}} \times K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q) \rightarrow \mathbb{Z} \quad (29)$$

via the isomorphism χ . Using the Kazhdan-Lusztig-Vogan algorithm, this second pairing is shown to be a simple pairing between irreducible perverse sheaves $P(\xi)$ and characters of irreducible representations $\pi(\xi)$ for $\xi \in \Xi({}^\vee G^\Gamma)^Q$ ((11.2)(e), Theorem 15.12 and Theorem 26.2 [ABV92]).

We are now in the position to describe endoscopic lifting in the standard case, *i.e.* when ${}^\vee \vartheta^\Gamma$ is trivial. It is essentially given by the restriction homomorphism

$$\epsilon^* : K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q) \rightarrow K\mathcal{C}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}) \quad (30)$$

which is defined by the inverse image functor on equivariant constructible sheaves (Proposition 7.18 and (7.19)(d) [ABV92]). The usual notion of endoscopic lifting between linear combinations of characters may be recovered by combining ϵ^* with the perfect pairing (28) as follows. Let η_H be a strongly stable complex linear combination of characters of representations in $\Pi^{z_H}(H/\mathbb{R})_{\hat{Q}_H}$ (*cf.* Definition 18.6 and Definition 26.13 [ABV92]). Then (28) allows us to view η_H as a complex-valued homomorphism on $K\mathcal{C}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H})$ (Theorem 26.2 [ABV92]). Taken this way, endoscopic lifting is defined by ϵ_* , where

$$\epsilon_* \eta_H(F_G) = \langle \epsilon_* \eta_H, F_G \rangle = \langle \eta_H, \epsilon^* F_G \rangle, \quad F_G \in K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q) \quad (31)$$

(Definition 26.3 and Definition 26.18 [ABV92]). One may regard $\epsilon_*\eta_H$ as a homomorphism on $K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ or, somewhat more traditionally, as a formal complex combination of characters of representations in $\Pi^z(G/\mathbb{R})_{\hat{Q}}$.

We have just described endoscopic lifting in terms of constructible sheaves and the pairing (28). In (31) one should regard η_H as a linear combination of characters of standard representations. We could equally well have described endoscopic lifting in terms of perverse sheaves and pairing (29). To do this, one should regard η_H as a linear combination of irreducible characters.

5.4 Twisted endoscopic lifting

We now consider *twisted* endoscopic lifting, *i.e.* when ${}^\vee\vartheta^\Gamma$ is a non-trivial outer automorphism. In the twisted case not all irreducible elements of $K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$ are relevant in (30). This is described in 25 [ABV92], although in much less detail than the case of standard endoscopy. For consistency with 25 [ABV92], set $\sigma = \text{Int}(s) \circ {}^\vee\vartheta^\Gamma$, keeping in mind that this is an automorphism of order $m < \infty$. We note that the automorphism σ acts on ${}^\vee G^Q$ (through a unique lift). Furthermore it acts on $X({}^\vee G^\Gamma)$ ((14)) in a manner that is compatible with the ${}^\vee G^Q$ -action on $X({}^\vee G^\Gamma)$ ((25.1)(b) [ABV92]). It follows that σ permutes the ${}^\vee G^Q$ -orbits of $X({}^\vee G^\Gamma)$, and induces an isomorphism $A_x^{loc,Q} \mapsto A_{\sigma \cdot x}^{loc,Q}$ for any $x \in X({}^\vee G^\Gamma)$. We first provide an example of the type of sheaves which are relevant to (30) in twisted endoscopy.

Example 5.8. Suppose $x \in X({}^\vee G^\Gamma)$ and $\sigma(x) = x$. Suppose further that $S = {}^\vee G^Q \cdot x$ and $\tau = \sum_{j=1}^k \tau_j$ is a sum of irreducible representations of $A_x^{loc,Q}$. The automorphism σ induces an automorphism of $A_x^{loc,Q}$. We suppose that $\tau \circ \sigma \cong \tau$. Under these circumstances τ extends to a representation of $A_x^{loc,Q} \rtimes \langle \sigma \rangle$, where $\tau(\sigma)$ is an intertwining operator between $\tau \circ \sigma$ and τ . Note that the choice for $\tau(\sigma)$ is not canonical as $\tau(\sigma)$ may be replaced by $\zeta\tau(\sigma)$ for any m th root of unity $\zeta \in \mathbb{C}^\times$.

We wish to translate this setup into the realm of constructible sheaves. Let $\xi_j = (S, \tau_j)$. The pair (S, τ) is equivalent to a ${}^\vee G^Q$ -equivariant local system $\oplus_{j=1}^k \mathcal{V}_{\xi_j}$ on S ((7.4) and Lemma 7.3 (c) [ABV92]). The automorphism σ passes to an automorphism of $\oplus_{j=1}^k \mathcal{V}_{\xi_j}$ in the guise of $\tau(\sigma)$, a linear isomorphism of the stalks. This automorphism of $\oplus_{j=1}^k \mathcal{V}_{\xi_j}$ further passes to an automorphism of the ${}^\vee G^Q$ -equivariant constructible sheaf $\oplus_{j=1}^k \mu(\xi_j)$ through the same isomorphism of the stalks. We abusively denote the isomorphism of $\oplus_{j=1}^k \mu(\xi_j)$ by $\tau(\sigma)$. The intertwining property of $\tau(\sigma)$ above is tantamount to a compatibility condition with the σ -action on ${}^\vee G^Q$. The pairs $(\oplus_{j=1}^k \mu(\xi_j), \tau(\sigma))$, become the principal objects in twisted endoscopy.

A slight generalization of the sheaves of Example 5.8 is given in (25.7) [ABV92]. There the objects of the category $\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ are pairs (C, σ_C) , in which C is a ${}^\vee G^Q$ -equivariant constructible sheaf, and σ_C is a finite-order automorphism of C compatible with the σ -action on ${}^\vee G^Q$ and $X({}^\vee G^\Gamma)$. The compatibility condition is natural, but a bit unwieldy to state. As it is not given

in [ABV92], we provide it here: Given any $x \in X({}^\vee G^\Gamma)$, $S = {}^\vee G^Q \cdot x$ and stalk C_x , there is a representation $\tau_S^{loc}(C)$ of $A_S^{loc, Q}$ with space C_x , obtained by restricting C to S (Corollary 23.3 [ABV92]). The compatibility condition on the morphism $\sigma_C : C \rightarrow C$ is that it induces a linear isomorphism $C_x \rightarrow C_{\sigma \cdot x}$ intertwining $\tau_S^{loc}(C)$ with $\tau_{\sigma(S)}^{loc}(C)$ for every x . In the twisted case the restriction homomorphism (30) of standard endoscopy is replaced with

$$\epsilon^* : KC(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma) \rightarrow KC(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma). \quad (32)$$

Despite appearances, the Grothendieck group $KC(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma)$ on the right differs only superficially from $KC(X({}^\vee H^\Gamma), {}^\vee H^{Q_H})$. Indeed, σ acts trivially on all objects defined from ${}^\vee H$ (cf. Definition 5.1). In consequence, all of the intertwining operators on stalks may be identified with their eigenvalues, which are m th roots of unity. This leads to an isomorphism

$$KC(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma) \cong KC(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}) \otimes \mathbb{Z}[U_m] \quad (33)$$

where $\mathbb{Z}[U_m] \subset \mathbb{C}$ is the group algebra of the m th roots of unity (cf. (25.7)(e) [ABV92]).

It is worth reflecting on the relationship between (30) and (32). In the case of standard endoscopy, σ is simply $\text{Int}(s)$, and $\sigma \cdot x \in S = {}^\vee G^Q \cdot x$ for every $x \in X({}^\vee G^\Gamma)$. The action of σ falls under the umbrella of the equivariance conditions on $C \in \mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q)$. In particular the equivariance produces an identification of stalks $C_x = C_{\sigma \cdot x}$, and the identity map on C_x is the canonical choice of intertwining operator between $\tau_S^{loc}(C)$ and $\tau_{\sigma(S)}^{loc}(C)$. In other words, one may define σ_C by taking it to be the identity map on all stalks. Roots of unity still play a role in the definition of $\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ as in Example 5.8, but they play a superficial role as in $\mathcal{C}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma)$. In summary, for standard endoscopy (32) is of the form

$$\epsilon^* : KC(X({}^\vee G^\Gamma), {}^\vee G^Q) \otimes \mathbb{Z}[U_m] \rightarrow KC(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma) \otimes \mathbb{Z}[U_m], \quad (34)$$

where ϵ^* is the identity map on U_m . This is the trivial extension of (30) to the tensor products with $\mathbb{Z}[U_m]$.

By contrast, in the case of twisted endoscopy, σ is an outer automorphism and the equivariance of C does not necessarily allow us to identify a stalk C_x with $C_{\sigma \cdot x}$. Even in the case that $\sigma \cdot x = x$ as in Example 5.8 there need not exist a canonical choice of intertwining operator on C_x .

In order to make a connection with Arthur packets in the next section, it is preferable to express twisted endoscopic lifting using perverse sheaves rather than constructible sheaves. The obvious extension of the foregoing discussion to chain complexes of equivariant constructible sheaves is defined by specifying that the sheaves in each degree of a complex belong to $\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$. One defines the category of perverse sheaves $\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ using the same principle. We call an object in $\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ a *twisted perverse sheaf*. Such an object is a pair (P, σ_P) , where $P = P^\bullet$ is a constructible complex (in the derived category and with additional structure), and σ_P is an automorphism

of P . It is easily verified that σ_P induces an automorphism of $H^i P$ for all $i \in \mathbb{Z}$. In consequence, one may extend (27) to an isomorphism

$$\chi : K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma) \rightarrow K\mathcal{C}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma) \quad (35)$$

in which $\chi(\sigma_P)$ is the induced morphism on $\oplus_{i \in \mathbb{Z}} H^i P$. We abusively denote the restriction homomorphism

$$\epsilon^* : K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma) \rightarrow K\mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma)$$

as in (32). The codomain here may be simplified as in (33).

Our next step is to describe twisted endoscopic lifting as a map on stable virtual characters along the lines of (31). Rather than working with arbitrary stable virtual characters η_H as in (31) one may work, without loss of generality, with a specific family of strongly stable virtual characters η_S^{mic, Q_H} , where S runs over all ${}^\vee H$ -orbits of $X({}^\vee H^\Gamma)$ (Corollary 19.16 [ABV92]). The definition of η_S^{mic, Q_H} rests on deep results in microlocal geometry which we shall not attempt to sketch in any detail. We shall settle for a peripheral description of η_S^{mic, Q_H} in terms of representations $\tau_S^{mic}(P)$, which are analogues of the representations $\tau_S^{loc}(C)$ mentioned earlier. The representation $\tau_S^{loc}(C)$ of A_S^{loc, Q_H} defines the local system equal to the restriction of $C \in \mathcal{C}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H})$ to S . Similarly, given $P \in \mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H})$ there exists a representation $\tau_S^{mic}(P)$ of the *equivariant micro-fundamental group* A_S^{mic, Q_H} which defines a ${}^\vee H^{Q_H}$ -equivariant local system $Q^{mic}(P)$ on a space determined by ${}^\vee H^{Q_H}$ and $X({}^\vee H^\Gamma)$ ((24.1), Definition 24.7, Theorem 24.8 and Corollary 24.9 [ABV92]). Passing over the sophisticated theory underlying these objects, we may write

$$\eta_S^{mic, Q_H} = \sum_{\xi \in \Xi({}^\vee H)^{Q_H}} e(\xi)(-1)^{\dim S_\xi - \dim S} \dim \tau_S^{mic}(P(\xi)) \pi(\xi)$$

(Definition 19.13, Definition 19.15, Corollary 19.16 and Corollary 24.9 (a) [ABV92]). This is a finite sum in which $e(\xi) = \pm 1$ (Definition 15.8 [ABV92]), S_ξ is the ${}^\vee H$ -orbit in ξ ((7.4) [ABV92]), and $\pi(\xi)$ is an irreducible (character of a) representation in $\Pi^{z_H}(H/\mathbb{R})_{\hat{Q}_H}$ given by (10). More generally, for any $h \in A_S^{mic, Q_H}$ one may define a formal complex virtual representation

$$\eta_S^{mic, Q_H}(h) = \sum_{\xi \in \Xi({}^\vee H)^{Q_H}} e(\xi)(-1)^{\dim S_\xi - \dim S} \text{tr}(\tau_S^{mic}(P(\xi))(h)) \pi(\xi) \quad (36)$$

(Definition 26.8 [ABV92]). According to Lemma 26.9 [ABV92],

$$\langle \eta_S^{mic, Q_H}(h), P(\xi') \rangle = (-1)^{\dim S} \text{tr}(\tau_S^{mic}(P(\xi'))(h)), \quad \xi' \in \Xi({}^\vee H^\Gamma)^{Q_H} \quad (37)$$

using pairing (29).

We would like to define an extension of η_S^{mic, Q_H} to $A_S^{mic, Q_H} \times \langle \sigma \rangle$ in order to incorporate the twisted perverse sheaves of $\mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma)$. Suppose $\xi \in \Xi({}^\vee H^\Gamma)^{Q_H}$ and $(P(\xi), \sigma_{P(\xi)}) \in \mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma)$. We first note that

$\sigma_{P(\xi)}$ may be identified with an m th root of unity. This identification is the essence of the notion of “eigenobjects” on page 272 [ABV92], which goes as follows: Since σ fixes all $x \in X({}^\vee H^\Gamma)$, the automorphism $\sigma_{P(\xi)}$ must induce a linear automorphism on each stalk $H^i P(\xi)_x$. Let $\zeta \in \mathbb{C}$ be an eigenvalue on some stalk $H^i P(\xi)_x$ and let $\zeta_{P(\xi)}$ be the automorphism of $P(\xi)$ given by multiplication by ζ on every stalk. Clearly, $\zeta \in U_m$ and $\ker(\sigma_{P(\xi)} - \zeta_{P(\xi)})$ is a non-zero perverse subsheaf of the irreducible sheaf $P(\xi)$. Hence, $\sigma_{P(\xi)} = \zeta_{P(\xi)}$.

The next step in the extension of η_S^{mic, Q_H} is the requisite extension of $\tau_S^{mic}(P(\xi))$ to $A_S^{mic, Q_H} \times \langle \sigma \rangle$. The ${}^\vee H$ -orbit S specifies special stalks of the local system $Q^{mic}(P(\xi))$ (Lemma 24.3 [ABV92]). We choose one, which we cryptically denote by $Q^{mic}(P(\xi))_{x, \nu}$ (x belongs to S , see Definition 24.7 [ABV92]). Recall that the local system $Q^{mic}(P(\xi))$ corresponds to the representation $\tau_S^{mic}(P(\xi))$. It is a representation of A_S^{mic, Q_H} on $Q^{mic}(P(\xi))_{x, \nu}$. The automorphism $\sigma_{P(\xi)}$ induces an automorphism of $Q^{mic}(P(\xi))_{x, \nu}$ ((25.1)(h), (24.10)(b) and Definition 24.11 [ABV92]). We denote this automorphism by $\tau_S^{mic}(P(\xi))(\sigma_{P(\xi)})$. This notation is consistent with (25.1)(j) [ABV92] and is appropriate, for $\tau_S^{mic}(P(\xi))(\sigma_{P(\xi)})$ is an operator intertwining $\tau_S^{mic}(P(\xi))$ with $\tau_S^{mic}(P(\xi)) \circ \sigma = \tau_S^{mic}(P(\xi))$. It is a consequence of $\sigma_{P(\xi)} = \zeta_{P(\xi)}$ that $\tau_S^{mic}(P(\xi))(\sigma_{P(\xi)})$ is multiplication by $\zeta \in U_m$. This yields the desired extension of $\tau_S^{mic}(P(\xi))$ to $A_S^{mic, Q_H} \times \langle \sigma \rangle$.

In completing our definition of $\eta_S^{mic, Q_H}(\sigma)$ we would like to arrange things so that some pairing with $K\mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma)$ results in a formula akin to (37). The natural objects to pair with $(P(\xi), \sigma_{P(\xi)})$ are objects of the form $(\pi(\xi'), \zeta')$, where $\zeta' \in U_m$ is to be interpreted as a self-intertwining operator of (any representative of) $\pi(\xi')$. The pairs $(P(\xi), \sigma_{P(\xi)})$ and $(\pi(\xi'), \zeta')$ may evidently be thought of as objects in

$$K\mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}, \sigma) = K\mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}) \otimes \mathbb{Z}[U_m]$$

and $K\Pi(H/\mathbb{R})_{\hat{Q}_H} \otimes \mathbb{Z}[U_m]$ respectively. We extend pairing (29) to a pairing

$$K\Pi(H/\mathbb{R})_{\hat{Q}_H} \otimes \mathbb{Z}[U_m] \times K\mathcal{P}(X({}^\vee H^\Gamma), {}^\vee H^{Q_H}) \otimes \mathbb{Z}[U_m] \rightarrow \mathbb{C}$$

by defining

$$\langle \pi(\xi') \otimes \zeta', P(\xi) \otimes \zeta \rangle = \zeta' \zeta \langle \pi(\xi'), P(\xi) \rangle. \quad (38)$$

We finally extend (36) to $h\sigma \in A_S^{mic, Q_H} \times \langle \sigma \rangle$ by defining

$$\eta_S^{mic, Q_H}(h\sigma) = \sum_{\xi \in \Xi({}^\vee H)^{Q_H}} e(\xi)(-1)^{\dim S_\xi - \dim S} \operatorname{tr}(\tau_S^{mic}(P(\xi))(h)) (\pi(\xi) \otimes 1).$$

Here, each $\pi(\xi) \otimes 1$ may be regarded as an irreducible character, or equivalently as an irreducible character twisted by the identity self-intertwining operator (6 [Art08]). Viewed in this way, twisted characters are paired with twisted perverse sheaves in (38), and we obtain

$$\begin{aligned} \langle \eta_S^{mic, Q_H}(h\sigma), (P(\xi), \sigma_{P(\xi)}) \rangle &= (-1)^{\dim S} \zeta \operatorname{tr}(\tau_S^{mic}(P(\xi'))(h)) \\ &= (-1)^{\dim S} \operatorname{tr}(\tau_S^{mic}(P(\xi'))(h\sigma_{P(\xi)})) \end{aligned} \quad (39)$$

when $\sigma_{P(\xi)}$ is identified with $\zeta \in U_m$ as above.

The reader has likely noticed that the introduction of twisted objects here for endoscopic groups appears artificial in the same way as it was in (34). The introduction of twisted objects is justified after comparison with twisting for ${}^\vee G^\Gamma$.

We now consider twisting for ${}^\vee G^\Gamma$. Under certain circumstances one may imitate the procedure leading up to (39) for ${}^\vee G^\Gamma$. Let us first assume that $S \subset X({}^\vee H^\Gamma)$ is an ${}^\vee H$ -orbit and

$$S' = X(\epsilon)(S) \subset X({}^\vee G^\Gamma) \quad (40)$$

is a ${}^\vee G$ -orbit (see (26)). By definition, S is σ -stable, and so S' is σ -stable. Suppose $\xi \in \Xi({}^\vee G^\Gamma)^Q$ is fixed by σ and that

$$(P(\xi), \sigma_{P(\xi)}) \in \mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma).$$

The ${}^\vee H$ -orbit S again specifies stalks of $Q^{mic}(P(\xi))$ and we fix one, $Q^{mic}(P(\xi))_{x,\nu}$ (x belongs to $S \subset S'$, see Definition 24.7 and Lemma 25.3 [ABV92]). As before we denote the representation of $A_{S'}^{mic,Q}$ on $Q^{mic}(P(\xi))_{x,\nu}$ by $\tau_{S'}^{mic}(P(\xi))$. The automorphism $\sigma_{P(\xi)}$ induces an automorphism $\tau_{S'}^{mic}(P(\xi))(\sigma_{P(\xi)})$ of $Q^{mic}(P(\xi))_{x,\nu}$ (cf. (25.1)(h) and (25.1)(i) [ABV92]). It is an operator intertwining $\tau_{S'}^{mic}(P(\xi))$ with $\tau_{S'}^{mic}(P(\xi)) \circ \sigma$, and yields an extension of $\tau_{S'}^{mic}(P(\xi))$ to $A_{S'}^{mic,Q} \rtimes \langle \sigma \rangle$. Unlike the construction for ${}^\vee H^\Gamma$, the intertwining operators need not be scalars.

The next step towards the construction of $\eta_{S'}^{mic,Q}(\sigma)$ is the definition of commensurate objects to pair with $(P(\xi), \sigma_{P(\xi)})$. The pairs $(\pi(\xi), \zeta) = \pi(\xi) \otimes \zeta$ above and the equivariance of Theorem 4.2 hint that these objects should be (equivalence classes of) ϑ^Γ -stable representations of strong real forms together with their intertwining operators. The m th roots ζ in the pairs $(\pi(\xi), \zeta)$ also indicate that the intertwining operators should in some way “remember” the order m of σ .

Lemma 5.9. *Suppose $\delta \in G^\Gamma - G$ is a strong real form and $g \in G$ such that $\vartheta^\Gamma \circ \text{Int}(g)(\delta) = \delta$, i.e. δ is equivalent to $(\vartheta^\Gamma)^{-1}(\delta)$. Then $(\vartheta^\Gamma \circ \text{Int}(g))^m = \text{Int}((\vartheta + \vartheta^2 + \cdots + \vartheta^m)(g))$ and $(\vartheta + \cdots + \vartheta^m)(g)$ belongs to $G(\mathbb{R}, \delta)$.*

Proof. Since

$$\sigma^m = (\text{Int}(s) \circ {}^\vee \vartheta^\Gamma)^m = \text{Int}((1 + {}^\vee \vartheta + \cdots + {}^\vee \vartheta^{m-1})(s)) \circ ({}^\vee \vartheta^\Gamma)^m$$

is the identity and $({}^\vee \vartheta^\Gamma)|_{{}^\vee G}^m = {}^\vee \vartheta^m$ is distinguished, it follows that ${}^\vee \vartheta^m$ is also the identity (16.5 [Hum72]). The compatibility of ${}^\vee \vartheta$ with the distinguished automorphism ϑ forces the two automorphisms to have the same order. Therefore ϑ^m is the identity and the first assertion follows. The second assertion follows directly from the definition of $G(\mathbb{R}, \delta)$ and

$$\delta = (\vartheta^\Gamma \circ \text{Int}(g))^m(\delta) = \text{Int}((\vartheta + \cdots + \vartheta^m)(g))(\delta)$$

□

Definition 5.10.

A *twisted representation of a strong real form* for $(G^\Gamma, \vartheta^\Gamma, \sigma)$ is a pair $((\pi, \delta), \mathcal{I})$ satisfying the following conditions.

1. (π, δ) is a representation of a strong real form of G^Γ (Section 2).
2. $\vartheta^\Gamma \cdot (\pi, \delta)$ is equivalent to (π, δ) , *i.e.* there exists $g \in G$ such that $(\vartheta^\Gamma) \circ \text{Int}(g)(\delta) = \delta$ and $\pi \circ \vartheta \circ \text{Int}(g)$ is infinitesimally equivalent to π (see (13)).
3. \mathcal{I} is a linear automorphism of the representation space of π such that for some $y \in G(\mathbb{R}, \delta)$ the operator $\mathcal{I}\pi(y)$ exhibits the equivalence of part 2 above, *i.e.*

$$\pi \circ \vartheta \circ \text{Int}(g) = \mathcal{I}\pi(y) \pi (\mathcal{I}\pi(y))^{-1}$$

(on the appropriate dense subspace).

4. $(\mathcal{I}\pi(y))^m = \pi((\vartheta + \dots + \vartheta^m)(g))$.

We note any other element of G producing the same equivalence in part 3 of this definition is of the form gx where $x \in G(\mathbb{R}, \delta)$, and for this element $\mathcal{I}\pi(yx)$ is an intertwining operator. The intertwining property then implies that

$$(\mathcal{I}\pi(yx))^m = (\mathcal{I}\pi(y)\pi(x))^m = \pi((\vartheta \circ \text{Int}(g) + \dots + (\vartheta \circ \text{Int}(g))^m)(x)) (\mathcal{I}\pi(y))^m$$

which may be verified to equal

$$\pi((\vartheta \circ \text{Int}(g) + \dots + (\vartheta \circ \text{Int}(g))^m)(x)) \pi((\vartheta + \dots + \vartheta^m)(g)) = \pi((\vartheta + \dots + \vartheta^m)(gx))$$

using Lemma 5.9. Thus, Definition 5.10 is well-defined.

Definition 5.11.

Two twisted representations of strong real forms $((\pi, \delta), \mathcal{I})$ and $((\pi', \delta'), \mathcal{I}')$ as in Definition 5.10 are *equivalent* if

1. (π, δ) is equivalent to (π', δ') , *i.e.* there exists $g_0 \in G$ such that $g_0 \delta g_0^{-1} = \delta'$ and $\pi \circ \text{Int}(g_0^{-1}) = A\pi' A^{-1}$ for some intertwining operator A .

- 2.

$$A^{-1}(\mathcal{I}\pi(y))^{-1} A \mathcal{I}' \pi'(y') = \pi'(g_0 g^{-1} \vartheta^{-1} (g_0^{-1}) g')$$

for $y, y', g, g' \in G$ as in 2 Definition 5.10.

We denote the set of equivalence classes of irreducible twisted representations of strong real forms for $(G^\Gamma, \vartheta, \sigma)$ by $\Pi(G/\mathbb{R}, \vartheta, \sigma)$.

The last equation in Definition 5.11 is motivated by the fact that the operator on the left intertwines π' with its composition under $\text{Int}(g_0 g^{-1} \vartheta^{-1} (g_0^{-1}) g')$. We omit the unpleasant verification that $g_0 g^{-1} \vartheta^{-1} (g_0^{-1}) g'$ lies in $G(\mathbb{R}, \delta')$ and that Definition 5.11 is well-defined. Definitions 5.10 and 5.11 have analogues for canonical projective representations of strong real forms (Definition 10.3

[ABV92]). We omit the details once again and write $\Pi^z(G/\mathbb{R}, \vartheta, \sigma)$ for the equivalence classes of type z . The subset of type \hat{Q} is denoted by $\Pi^z(G/\mathbb{R}, \vartheta, \sigma)_{\hat{Q}}$.

The elements of $\Pi^z(G/\mathbb{R}, \vartheta, \sigma)_{\hat{Q}}$ are the objects we wish to pair with $\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$. This presents us with the following problem: given σ -fixed $\xi', \xi \in \Xi({}^\vee G^\Gamma)^Q$ and (representatives) $(\pi(\xi'), \mathcal{I})$, $(P(\xi), \sigma_{P(\xi)})$ in $\Pi^z(G/\mathbb{R}, \vartheta, \sigma)_{\hat{Q}}$ and $\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ respectively, how does one define

$$\langle (\pi(\xi'), \mathcal{I}), (P(\xi), \sigma_{P(\xi)}) \rangle \in \mathbb{C} \quad (41)$$

in a canonical fashion. For endoscopic groups, the pairing (38) was defined using the canonical base point $(\pi(\xi), 1)$ with trivial self-intertwining operator, and the canonical base point $(P(\xi), \sigma_{P(\xi)})$ with $\sigma_{P(\xi)}$ acting trivially. If one could similarly choose a canonical intertwining operator $\mathcal{I}_{\xi'}$ and a canonical automorphism σ_ξ in (41), then all other relevant intertwining operators and automorphisms would differ from these by roots of unity in U_m and we could set

$$\langle (\pi(\xi'), \zeta' \mathcal{I}_{\xi'}), (P(\xi), \zeta \sigma_\xi) \rangle = \zeta' \zeta \langle \pi(\xi'), P(\xi) \rangle, \quad \zeta', \zeta \in U_m. \quad (42)$$

This would constitute a canonical pairing

$$K\Pi^z(G/\mathbb{R}, \vartheta, \sigma)_{\hat{Q}} \times K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma) \rightarrow \mathbb{C}.$$

We are not prepared to solve the general problem of choosing a canonical intertwining operator $\mathcal{I}_{\xi'}$ for $\pi(\xi')$ as indicated. It is solved in the examples of twisted endoscopy for $G = \mathrm{GL}_n$, where Q is trivial, in 2.1 [Art13], 8 [AMR] and 3.2 [Mok15]. In these examples one only encounters the quasisplit forms of general linear groups, and therefore has recourse to Whittaker data to normalize intertwining operators. Perhaps there is a simple resolution of the problem for arbitrary quasisplit groups, but this is less likely for all strong inner forms.

The problem of choosing a canonical automorphism σ_ξ for $P(\xi)$ does have a simple resolution using the map χ as in (27). It is known that the irreducible constructible sheaf $\mu(\xi)$ occurs in $\chi(P(\xi))$ with multiplicity one ((7.11)(e) [ABV92]). In consequence, an automorphism $\chi(\sigma_{P(\xi)})$ restricts to an automorphism of $\mu(\xi)$. The latter automorphism induces a self-intertwining operator of the character $\tau_{S_\xi}^{loc}(\mu(\xi))$ as in the discussion preceding (32). As such, it is a root of unity in U_m . We choose the canonical automorphism σ_ξ of $P(\xi)$ to be the one for which this root of unity is $1 \in U_m$.

At last we are able to describe twisted endoscopic lifting, at least under the assumption that the canonical intertwining operators in the definition of (41) are defined. Under this assumption we define

$$\eta_{S'}^{mic, Q}(\sigma) = \sum_{\xi \in \Xi({}^\vee G)^Q} e(\xi)(-1)^{\dim S_\xi - \dim S'} \mathrm{tr}(\tau_{S'}^{mic}(P(\xi))(\sigma_\xi)) (\pi(\xi), \mathcal{I}_\xi), \quad (43)$$

where $\tau_{S'}^{mic}(P(\xi))(\sigma_\xi) = 0$ when $\sigma \cdot \xi \neq \xi$. This is to be regarded as a complex combination of twisted characters. Using pairing (41), one may compute as in (39) that for $(P(\xi), \sigma_{P(\xi)}) \in K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ we have

$$\langle \eta_{S'}^{mic, Q}, (P(\xi), \sigma_{P(\xi)}) \rangle = (-1)^{\dim S'} \mathrm{tr}(\tau_{S'}^{mic}(P(\xi))(\sigma_{P(\xi)})).$$

The twisted endoscopic lifting identity of Theorem 25.8 [ABV92] may now be read as

$$\langle \eta_{S'}^{mic, Q}(\sigma), (P(\xi), \sigma_{P(\xi)}) \rangle = \langle \eta_S^{mic, Q_H}(\sigma), \epsilon^*(P(\xi), \sigma_{P(\xi)}) \rangle \quad (44)$$

for all $(P(\xi), \sigma_{P(\xi)}) \in K\mathcal{P}(X({}^\vee G^\Gamma), {}^\vee G^Q, \sigma)$ (see (25.1)(j) and (24.10) [ABV92]). This identity defines the twisted endoscopic lifting map ϵ_* from the complex vector space generated by $\eta_S^{mic, Q_H}(\sigma)$ to the vector space generated by $\eta_{S'}^{mic, Q}$ via $\epsilon_* \eta_S^{mic, Q_H}(\sigma) = \eta_{S'}^{mic, Q}(\sigma)$ (cf. (31)).

6 Twisted endoscopy for GL_N and Arthur packets of classical groups

In this concluding section we illustrate how the twisted endoscopic identity (44) is applicable to Arthur packets as defined in [ABV92]. We shall do this in the framework of Examples 5.2 and 5.5 which were chosen to match the endoscopic classification of Arthur ([Art13]).

We begin by recalling the definitions leading up to Arthur packets. In these definitions ${}^\vee G^\Gamma$ is an arbitrary weak E-group. An *Arthur parameter* (or *A-parameter*) is a homomorphism

$$\psi : W_{\mathbb{R}} \times SL_2 \rightarrow {}^\vee G^\Gamma$$

such that $\psi|_{W_{\mathbb{R}}}$ is a tempered (*i.e.* bounded) L-parameter and $\psi|_{SL_2}$ is holomorphic.

Let ψ be an A-parameter. There is an *associated L-parameter*

$$\phi_\psi : W_{\mathbb{R}} \rightarrow {}^\vee G^\Gamma$$

defined by

$$\phi_\psi(w) = \psi \left(w, \begin{bmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{bmatrix} \right), \quad w \in W_{\mathbb{R}}.$$

There is also an associated *Arthur component group* A_ψ , which is the component group of the centralizer in ${}^\vee G$ of the image of ψ . These definitions are due to Arthur (§4 [Art89]).

The first connection with the theory of [ABV92] is that the Arthur component group A_ψ is equal to an equivariant micro-fundamental group $A_{S_\psi}^{mic, Q}$. Here, Q is trivial and $S_\psi \subset X({}^\vee G^\Gamma)$ is the ${}^\vee G$ -orbit³ corresponding to ϕ_ψ via Proposition 6.17 [ABV92] (see also Definition 24.15 and Proposition 22.9 [ABV92]). For this reason we assume from now on that Q is trivial.

Recall from the previous section that for every $\xi \in \Xi({}^\vee G^\Gamma)^Q$ there is a representation $\tau_{S_\psi}^{mic}(P(\xi))$ of $A_{S_\psi}^{mic, Q} = A_\psi$. Suppose that the hypotheses of

³The orbit S_ψ here is not to be confused with the centralizer S_ψ in [Art89] or [Art13].

Theorem 4.2 are satisfied so that (10) holds. Then the *Arthur packet* (or *A-packet*) of ψ [ABV92] may be described as the set of $\pi(\xi) \in \Pi^z(G/\mathbb{R})_{\hat{Q}}$ whose multiplicity in

$$\eta_\psi = \eta_{S_\psi}^{mic, Q} = \sum_{\xi \in \Xi({}^\vee G^\Gamma)^Q} e(\xi)(-1)^{\dim S_\xi - \dim S_\psi} \dim \tau_S^{mic}(P(\xi)) \pi(\xi)$$

is positive (*cf.* Definition 19.13, Definition 19.15, Corollary 19.16, Definition 22.6, Corollary 24.9 (a) [ABV92]). In other words, the A-packet of ψ is

$$\Pi^z(G/\mathbb{R})_{\hat{Q}, \psi} = \{\pi(\xi) : \tau_{S_\psi}^{mic}(P(\xi)) \neq 0\}. \quad (45)$$

We should emphasize that the assumption of Q being trivial in our description results in A-packets which are potentially smaller than the extended A-packets defined for $Q = \pi_1({}^\vee G)^{alg}$ in Definition 22.6 [ABV92]. To be honest, Adams-Barbasch-Vogan do not even give a name to the sets in (45), although they play a key role in endoscopy (*cf.* Definition 26.8 [ABV92]).

Henceforth, we work under the assumptions of Examples 5.2 and 5.5. In particular, $G = \mathrm{GL}_N$ and the endoscopic group H is a product of symplectic or orthogonal groups. Our task is to reveal the relationship between the twisted endoscopic transfer identity (44) and the A-packets we have just defined.

We fix an A-parameter $\psi_H : W_{\mathbb{R}} \times \mathrm{SL}_2 \rightarrow {}^\vee H^\Gamma$ and define $\eta_{\psi_H} = \eta_{S_{\psi_H}}^{mic, Q_H}$. Recall that S_{ψ_H} is the ${}^\vee H$ -orbit corresponding to ϕ_{ψ_H} via Proposition 6.17 [ABV92], and Q_H is trivial. More generally, we set $\eta_{\psi_H}(h\sigma) = \eta_{S_{\psi_H}}^{mic, Q_H}(h\sigma)$ as in (39). The virtual representation $\eta_{\psi_H}(\sigma)$ is to appear on the right-hand side of (44).

For the left-hand side, set $\psi_G = \epsilon \circ \psi_H$ so that ψ_G is an A-parameter for G . It is a simple exercise to show that the ${}^\vee G$ -orbit S_{ψ_G} corresponding to $\phi_{\psi_G} = \epsilon \circ \phi_{\psi_H}$ is equal to the ${}^\vee G$ -orbit of $X(\epsilon)(S_{\psi_H})$ (*cf.* (40), (26)). Let $\eta_{\psi_G}(\sigma) = \eta_{S_{\psi_G}}^{mic, Q}(\sigma)$ as in (43). Recall that in this definition we are making use of a canonical normalization of intertwining operators (2.1 [Art13], 8 [AMR]).

The twisted endoscopic transfer identity (44) is now in the form

$$\langle \eta_{\psi_G}(\sigma), (P(\xi), \sigma_{P(\xi)}) \rangle = \langle \eta_{\psi_H}(\sigma), \epsilon^*(P(\xi), \sigma_{P(\xi)}) \rangle. \quad (46)$$

This identity is a sheaf-theoretic analogue of Arthur's equation (2.2.3) [Art13] in which $\eta_{\psi_H}(\sigma)$ plays the role of the stable linear form defining an A-packet.

Equation (46) is significant in its own right, for it yields information about the elusive $\eta_{\psi_H}(\sigma)$ in terms of $\eta_{\psi_G}(\sigma)$, which is better understood. Let us investigate a specific example. The reader should revisit Examples 5.2 and 5.5 for the background.

Example 6.1. Take $N = 2$, $G = \mathrm{GL}_2$, $N_O = 1$ and $N'_S = 1$. Then $\sigma = {}^\vee \vartheta = \vartheta$ has order two, ${}^\vee H^\Gamma = \mathrm{Sp}_2 \times \Gamma = \mathrm{SL}_2 \times \Gamma$, and $H = \mathrm{SO}_3 \cong \mathrm{PGL}_2$. The second invariants of both ${}^\vee G^\Gamma$ and ${}^\vee H^\Gamma$ are trivial. As is customary, we ignore the copy of Γ in the previous direct products. A-parameters $\psi_H : W_{\mathbb{R}} \times \mathrm{SL}_2 \rightarrow \mathrm{SL}_2 \times \Gamma$ may be divided into two types: those whose restriction $(\psi_H)|_{\mathrm{SL}_2}$ is trivial and

those whose restriction $(\psi_H)|_{\mathrm{SL}_2}$ is the identity map. The former case is the case of tempered A-parameters. In this case the A-packets reduce to L-packets and both η_{ψ_H} and η_{ψ_G} are well-known for any groups (page 19 [ABV92]).

In the latter case $\psi_H(W_{\mathbb{R}})$ is central in SL_2 and it is harmless to assume that it is trivial. One may compute that

$$\phi_{\psi_H}(z) = \phi_{\psi_G}(z) = \begin{bmatrix} |z\bar{z}|^{1/2} & 0 \\ 0 & |z\bar{z}|^{-1/2} \end{bmatrix}, \quad z \in \mathbb{C}^\times$$

and $\phi_{\psi_H}(j) = \phi_{\psi_G}(j) = I$. The L-packet of ϕ_{ψ_G} (of type \hat{Q}) is the trivial representation of $\mathrm{GL}(2, \mathbb{R})$. Similarly the L-packet of ϕ_{ψ_H} (of type \hat{Q}_H) is the trivial representation of $\mathrm{PGL}(2, \mathbb{R})$.

It is generally true that the A-packet of an A-parameter contains the L-packet of its associated L-parameter (Lemma 19.14 (b) [ABV92]). By a comparison with the A-packets of Arthur for general linear groups (pages 24-25 and (2.2.1) [Art13]), one would expect the A-packet of ψ_G to be equal to the L-packet of ϕ_{ψ_G} . This is indeed true, although we see no simple explanation for this fact, even for $N = 2$. The explanation given in [ABV92] relies on ψ_G being *unipotent* (Definition 27.1, Theorem 27.18 (d), Example 19.17 [ABV92]). The explanation is equally valid for $H = \mathrm{PGL}_2$ and so we may write

$$\Pi(G/\mathbb{R})_{\hat{Q}, \psi_G} = \{\mathbf{1}_{\mathrm{GL}(2, \mathbb{R})}\}, \quad \Pi(H/\mathbb{R})_{\hat{Q}_H, \psi_H} = \{\mathbf{1}_{\mathrm{PGL}(2, \mathbb{R})}\}.$$

Incidentally, the triviality of Q and Q_H manifests itself here in the absence of the trivial representations of the other (strong) real forms of G and H .

Even though the A-packets in this example are transparent, it is still instructive to consider (46). On the left of (46) we have

$$\eta_{\psi_G}(\sigma) = e(\xi) \operatorname{tr} \left(\tau_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi))(\sigma_\xi) \right) (\pi(\xi), \mathcal{I}_\xi).$$

As we know that the A-packet of ψ_G is a singleton, the sum over $\xi \in \Xi({}^\vee G^\Gamma)^Q$ in (43) is reduced to a single index $\xi = (S_{\psi_G}, \mathbf{1})$. The ${}^\vee G$ -orbit S_{ψ_G} has been described earlier. The second term $\mathbf{1}$ denotes the trivial, and only, representation of the component group $A_{S_{\psi_G}}^{\mathrm{loc}, Q} = {}^\vee G_{\phi_{\psi_G}} / ({}^\vee G_{\phi_{\psi_G}})_0 = \{1\}$. It is an immediate consequence of Definition 15.8 [ABV92] that $e(\xi) = 1$. According to Corollary 24.9 (b) and Proposition 23.2 (b) [ABV92], the representation $\tau_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi))$ is equal to $\tau_{S_{\psi_G}}^{\mathrm{loc}}(\mu(\xi)) = \mathbf{1}$. The intertwining operator $\tau_{S_{\psi_G}}^{\mathrm{mic}}(P(\xi))(\sigma_\xi) = \pm 1$ has been normalized to equal 1. Finally, a straightforward verification of 2.1 [Art13] shows that the self-intertwining operator \mathcal{I}_ξ of $\pi(\xi) = \mathbf{1}_{\mathrm{GL}(2, \mathbb{R})}$ is normalized to equal 1. In short, $\eta_{\psi_G}(\sigma) = (\mathbf{1}_{\mathrm{GL}(2, \mathbb{R})}, 1)$.

The pairing on the left-hand side of (46) is taken with twisted perverse sheaves $(P(\xi'), \sigma_{\xi'})$, for any $\xi' = (S_{\xi'}, \tau_{\xi'}) \in \Xi({}^\vee G^\Gamma)^Q$. Since the component groups $A_{S_{\xi'}}^{\mathrm{loc}}$ are all trivial, the representations $\tau_{\xi'}$ are all trivial. This implies that the local system $\mathcal{V}_{\xi'}$ on each ${}^\vee G$ -orbit $S_{\xi'}$ is a constant sheaf.

Let us describe the orbits $S_{\xi'}$. The definition of the pairing allows us to restrict our attention to the orbits contained in some variety $X(\mathcal{O}, {}^\vee G^\Gamma)$. As it

happens, $\mathbf{1}_{\mathrm{GL}(2, \mathbb{R})}$ corresponds to

$$((y, \mathcal{F}(\lambda)), \tau) = \left(\left(\begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \mathcal{F} \left(\begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix} \right) \right), \mathbf{1} \right)$$

under the local Langlands Correspondence (6). Taking $\mathcal{O} = {}^\vee G \cdot \lambda$, Proposition 6.16 [ABV92] describes the $S_{\xi'}$ as ${}^\vee G_y$ -orbits of the complete flag variety of GL_2 . Clearly, the centralizer ${}^\vee G_y$ is the diagonal subgroup of GL_2 , and it is well-known that the flag variety is isomorphic to the projective line \mathbb{P}^1 . There are three resulting orbits in \mathbb{P}^1 , namely $\{0\}$, $\{\infty\}$ and the open orbit. Let us label these orbits as S_1 , S_2 and S_3 respectively. It is safe to identify these orbits with their corresponding ${}^\vee G$ -orbits on $X(\mathcal{O}, {}^\vee G^\Gamma)$ (Proposition 7.14 [ABV92]). We may also identify S_k with $\xi_k = (S_k, \mathbf{1})$, for $k = 1, 2, 3$. By construction $\pi(\xi_1) = \mathbf{1}_{\mathrm{GL}(2, \mathbb{R})}$. In consequence one may compute the pairing (42)

$$\langle (\eta_{\psi_G}(\sigma), (P(\xi_k), \sigma_{\xi_k})) \rangle = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases} \quad (47)$$

(cf. Theorem 1.24 [ABV92]).

The perverse sheaves $P(\xi_k)$ are determined by the closure relations among S_1 , S_2 and S_3 . Since the closure $\overline{S_{\xi_k}}$ of the orbit S_{ξ_k} is non-singular, the perverse sheaf $P(\xi_k)$ is the constant sheaf on $\overline{S_{\xi_k}}$ concentrated in degree $-\dim S_k$ (Lemma 4.3.2 [BBD82]).

Moving to the right-hand side of (46), we observe that there is an identical description of the relevant ${}^\vee H$ -orbits of $X({}^\vee H^\Gamma)$ in terms of the three orbits of the complete flag variety of ${}^\vee H = \mathrm{SL}_2$ (which is again isomorphic to \mathbb{P}^1). We denote the three orbits by $S_{H,1}$, $S_{H,2}$, and $S_{H,3}$ to distinguish them from the earlier ${}^\vee G$ -orbits. As earlier, the perverse sheaf $P(\xi_{H,k})$ is the constant sheaf on $\overline{S_{\xi_{H,k}}}$ concentrated in degree $-\dim S_{H,k}$. The restriction map ϵ^* is nearly negligible and has the obvious effect on the perverse sheaves: $\epsilon^*(P(\xi_k), \sigma_{\xi_k}) = (P(\xi_{H,k}), \sigma_{\xi_{H,k}})$. In consequence, identities (46) and (47) combine to give us

$$\langle (\eta_{\psi_H}(\sigma), (P(\xi_{H,k}), \sigma_{\xi_{H,k}})) \rangle = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases} \quad (48)$$

It follows in turn that $\eta_{\psi_H}(\sigma) = (\pi(\xi_{H,1}), 1) = (\mathbf{1}_{\mathrm{PGL}(2, \mathbb{R})}, 1)$ and $\Pi(H/\mathbb{R})_{\hat{Q}_H, \psi_H} = \{\mathbf{1}_{\mathrm{PGL}(2, \mathbb{R})}\}$ as expected.

In Example 6.1 we have delineated a method for computing $\eta_{\psi_H}(\sigma)$, and thereby the A-packet $\Pi^{z_H}(H/\mathbb{R})_{\hat{Q}_H, \psi_H}$, by using (46). It is possible to pursue this method for higher rank $G = \mathrm{GL}_N$, but there are notable impediments. We have already mentioned the difficulty in proving that the A-packet $\Pi(G/\mathbb{R})_{\hat{Q}, \phi_G}$ is a singleton when ψ_G is not unipotent.

Beyond this, one needs a description of the ${}^\vee G$ -orbits and ${}^\vee H$ -orbits occurring in the complete local parameters. If the infinitesimal character of the A-packet $\Pi(G/\mathbb{R})_{\hat{Q}, \psi_G}$ is regular then good descriptions for these orbits are given in terms of orbits on complete flag varieties ([Yam97], [Wys02]). If the

infinitesimal character is singular then additional work is required to describe these orbits in terms of orbits on partial flag varieties.

Finally, the perverse sheaves $P(\xi)$ are difficult to characterize when the closure of the orbit S_ξ is singular ([McG09]). One approach to such a characterization is through the map χ of (27) and its relationship to representation theory via the Kazhdan-Lusztig-Vogan algorithm ((7.11)(e) and Corollary 1.25 [ABV92]). Even if this approach does not provide an overt conceptual characterization, it is amenable to computation (using software such as the ATLAS of Lie groups and representations [Atl]).

Unfortunately this does not suffice, for we need a characterization of *twisted* perverse sheaves $(P(\xi), \sigma_\xi)$. The obvious course of action is to use the extended map χ of (35) together with a twisted version of the Kazhdan-Lusztig-Vogan algorithm ([LV14]). The details of this strategy should follow [AV15] and 19 [AvLTV]. One may describe the expected outcome of the strategy as follows. The map χ furnishes a decomposition of $(P(\xi), \sigma_\xi)$ as a virtual sum of irreducible twisted constructible sheaves. Likewise, an irreducible twisted representation may be decomposed as a virtual sum of twisted standard representations. The expected result is that there is a simple combinatorial formula which exhibits an equivalence between the two decompositions (*cf.* Corollary 1.25 [ABV92]).

This equivalence would allow one to recast the twisted endoscopic identity (46), replacing the twisted perverse sheaves with twisted constructible sheaves. The twisted constructible sheaves are simple to compute. Beyond this, the advantage of the alternative twisted endoscopic identity would be the ease of comparison with the twisted character identities defining the A-packets of Arthur ((8.3.4) [Art13]). The equality of the A-packets of (45) and the A-packets of [Art13] promises to follow from such a comparison.

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